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Ergodicity Conditions and Cesàro Limit Results for Marked Point Processes

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Abstract

In Palm theory it is very common to consider several distributions to describe the characteristics of the system. To study a stationary marked point process, the time-stationary distribution P and its event-stationary Palm distributions P_L^0 with respect to sets L of marks can all be used as starting point. When P is used, a modified, event-stationary version Q_L^0 of P_L^0 is defined as the limit of an obvious discrete-time Cesàro average. In a sense this modified Palm distribution is more natural than the ordinary one. When a Palm distribution $P_{L'}^0$ is taken as starting point, we can approximate another modified, **event**-stationary version of P_L^0 by considering discrete-time Cesàro averages and a modified, **time**-stationary version Q_L of P by considering continuous-time Cesàro averages. These and other limit results are corollaries of uniform limit theorems for Cesàro averaged functionals.

In essence, this paper presents a profound study of the relationship between $P, P_L^0, P_{L'}^0$, and modified versions of them, and their connections with ergodicity conditions and long-run averages of Cesàro type.

Keywords: Marked point process, Cesàro convergence, limit theorems, time-stationarity, event-stationarity, (modified) Palm distribution, ergodicity, pseudo-ergodicity, invariant σ-field, Radon-Nikodym density.

1 INTRODUCTION

Many problems in the study of series of events concern the relationship between event-stationary characteristics and time-stationary characteristics. In queueing theory, where the events are called arrival times, we mention Little's law, ASTA properties, and rate conservation laws; see, e.g., Baccelli and Brémaud [2]. In risk theory, talking about claim epochs instead of events, problems of this type appear when considering claim processes, risk processes, and ruin times; see, e.g., Asmussen and Schmidt [1] and Miyazawa and Schmidt [13]. The underlying theory for such problems is Palm theory for marked point processes (MPP's), where the points (events, occurrences) correspond to arrival times or claim epochs, and the marks to objects brought by the customers (e.g. service times) or claim sizes and/or types of the claims. In this theory the relationship is studied between the distribution P of the MPP Φ, representing the time-stationary characteristics, and a Palm distribution P^0, representing the event-stationary characteristics. One way to compare P and P^0 is to approximate the first when starting from the second, and vice versa. In this research we will consider approximations in terms of limit results for Cesàro averaged functionals of the form $\frac{1}{t}\int_0^t \mathbb{E}f(T_s\Phi)ds$ and $\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}f(\theta_i\Phi)$, and the relationship to ergodicity properties. Here T_s represents the shift to the time point s in $\mathbb{R}$, and θ_i the shift to the i'th occurrence.

Let P be the distribution of a time-stationary MPP Φ on $\mathbb{R}$ and let P_L^0 be its Palm distribution with respect to a set L of marks. Here time-stationarity means that the MPP has the same distribution seen from all time points in $\mathbb{R}$. A formal definition of P_L^0 follows below, but intuitively it is the conditional distribution of Φ given the occurrence of an "L-point" (an occurrence having its mark in L) in the origin. This intuitive definition is motivated by the local characterization of the Palm distribution as a limit of conditional probability measures. See Theorem 1.3.7 in Franken et al. [7] or Theorem 10 in Nieuwenhuis [18]. Inspired by the definition of P_L^0 in (2.1) and the inversion formula in (2.3), the relationship between P and P_L^0 is often (as in the unmarked case, see Nieuwenhuis [18]) also described by the following intuitive formulations:

P arises from P_L^0 by shifting the origin to a time point in $(-\infty, +\infty)$ chosen at random. (1)

P_L^0 arises from P by shifting the origin to an L -point chosen at random. (2)

A formalization of the intuitive random procedure in (1.1) is used for the *length-biased* *sampling* (LBS) procedure mentioned in Cox and Lewis [6] to derive relations between P and the Palm distribution. In the present context of MPP's this formalization would go like this. Starting from the situation that there is an L-point in the origin (i.e. P_L^0 is the ruling probability measure), the interval up to the r'th L-point is considered. Here r is very large. In this interval a time point is chosen **at random** and the origin is moved to it. It is argued that (as $r \to \infty$) the situation seen from this new position of the origin is described by P. The heuristic arguments used on page 61 of the last reference depend, however, heavily on whether a law of large numbers with degenerate limit holds for the sequence of interval lengths between the occurrences. The question arises if the formalization of (1.1) used in the LBS procedure is also applicable if the limit of the strong law is **non**degenerate.

One of the objectives of this research is to clarify the intuitive random procedures (1.1) and (1.2) for generating P and P_L^0 by choosing obvious formalizations. The formalizations of (1.1) and (1.2) are in terms of limit results for Cesàro averages. Note that the LBS procedure motivates the use of such averages for (1.1) because of the shift of the origin to a time point which is chosen **at random**. In Nieuwenhuis [18] it is proved that for (unmarked) point processes a formalization of (1.2) with Cesàro averages only leads to the Palm distribution if a weak ergodicity condition is satisfied. The generalization to marked point processes is, however, straightforward. Relation (54) and Theorem 7 in the above reference can be generalized and read as follows: When starting from P the distribution of the MPP seen from an L-point, chosen at random among the first n L-points, tends (as $n \to \infty$) in total variation to a distribution Q_L^0 which equals P_L^0 under a weak ergodicity condition. See Theorem 2.2 below. We believe that it is not the ordinary Palm distribution P_L^0 which fits (1.2). The *modified Palm distribution* Q_L^0 gives a better correspondence with this intuitive formulation. Literature is not so strict in these matters. Often an ergodicity assumption is made implicitly, to let intuition be true. In a sense even in Palm [19], where the theory was started up, it was (1.2) which motivated Palm to state his results. But ergodicity was not assumed. In many text books and papers on point processes, queueing theory and risk theory this problem is avoided by assuming ergodicity in advance, and by noting that sometimes a non-ergodic MPP can be considered as a mixture of two ergodic ones. However, an underlying mixture is not always well described. Furthermore, in the monograph Sigman [21] it is shown that for MPP's on the half line there is no need to make any ergodicity assumption for having a nice theory. See also Nieuwenhuis [16] and [18]. Similarly, the obvious formalization of the intuitive relationship between P and P_L^0 as described in (1.1) is only valid if a weak ergodicity condition holds. If this condition is not satisfied, it is not P which is described here but a *modified* time-stationary version Q_L of it. See Section 4.

Essentially, the above observations have strongly motivated this research. The main purpose of this paper is to present a profound study of the relationship between ergodicity, Palm distributions, modified Palm distributions, time-stationary distributions, and modified time-stationary distributions. We will consider two types of shifts, the time shifts $T_t, \quad t \in \mathbb{R}$, and, for each subset L of marks, the point shifts $\theta_{n,L}, \quad n \in \mathbb{Z}$. Here $T_t\Phi$ arises from Φ by shifting the origin to the time point t, while $\theta_{n,L}\Phi$ arises from Φ by shifting the origin to the n'th L-point, an occurrence with mark ik L. From the definition of time-stationarity mentioned above it is obvious that this notion expresses nothing but invariance of P under the family $\{T_t : t \in \mathbb{R}\}$. Similarly, the Palm distribution P_L^0 is invariant under the family $\{\theta_{n,L} : n \in \mathbb{Z}\}$ of shifts; we call this *event-stationarity* as in Sigman [21]. Each of the families of shifts induces its own invariant σ-field. Fortunately, these σ-fields turn out to be identical, see Lemmas 3.1 and 3.2, which makes the theory much easier. Ergodicity can be characterized in terms of laws of large numbers and conditional expectations on the invariant σ-field. The theory of these conditional expectations, treated in Section 3, is rather technical, but very important for the present research. We also present the notion of pseudo-L-ergodicity, which gives the weakest condition for the ordinary Palm distribution to be identical to the modified version. An example of a pseudo-L-ergodic MPP which is non-ergodic, will be considered in Section 3.

In Section 2 we formalize some of the notions mentioned above and give some preliminary results. Starting from P, we can approximate the modified Palm distribution Q_L^0 by Cesàro averages. See Theorem 2.2 which expresses a formalization of (1.2). The correspondent convergence even holds in total variation or, equivalently, uniform over all functions f with $|f| \le 1$. In Section 5 this result is generalized, yielding necessary and sufficient conditions for functions g, more general than the function which is identical to 1, to have this uniform convergence even over all functions f with $|f| \le g$. See Theorem 5.1 and Corollary 5.1. In Section 6 the distribution P is replaced by a Palm distribution $P_{L'}^0$, where L' is another nonempty set of marks. When starting from $P_{L'}^0$ the distribution of the MPP seen from an L-point, chosen at random among the first n L-points, tends uniformly to P_L^0 provided that a weak ergodicity condition is satisfied. In Section 4 a formalization of (1.1) is considered, so the roles of P and P_L^0 in Theorem 2.2 are interchanged: When starting from P_L^0 the distribution of the MPP seen from a position chosen at random between 0 and t tends in total variation to a modified time-stationary distribution Q_L (as $t \to \infty$) which equals P if a weak ergodicity condition is satisfied. Things can again be generalized by replacing the set of functions f with $|f| \leq 1$ in this uniform limit result by a more general set of functions f with $|f|$ bounded by a fixed function g. Necessary (and sufficient) conditions on g are formulated for the corresponding uniform convergence, see Theorem 4.1 and Corollary 4.1. Relations between Q_L, P, P_L^0 and Q_L^0 are derived. In Section 7 the theorems of Sections 4, 5 and 6 are applied. It is proved that, when starting from P_L^0 and P (or $P_{L'}^0$) respectively, P and P_L^0 can still be approximated uniformly by Cesàro means without assuming any ergodicity condition. Only the weights of the realizations of Φ have to be changed.

In our proofs we have to go from P_L^0 to P or from P to $P_{L'}^0$ several times. The method used to bridge these gaps (the"Radon-Nikodym approach", see Section 1 in Nieuwenhuis [18]), is a consequence of Theorem 2.1 below. This theorem represents the main tool for the approach in this research. Many of the results in this paper are beyond ergodicity. We need a rigorous, mathematical approach to prove them. We will, however, also use heuristic and intuitive arguments to motivate them.

As mentioned above, emphasis will be on MPP's on $\mathbb{R}$, the whole set of reals. So, when observing the MPP from a fixed time point in $\mathbb{R}$, we will not only have to deal with the presence and the future, but also with the past. In Glynn and Sigman [8] a theorem is proved for synchronous processes associated with a point process on the half line $[0, \infty)$, which is similar to Theorem 4.1 below giving uniform approximations (with Cesàro averaged functionals) of a time-stationary distribution when starting with an event-stationary distribution of Palm type. In the present research the mathematical setting in terms of MPP's on $\mathbb{R}$ is quite different from that in the above reference. Also the proofs differ. In the ergodic case, total variation limit results for Cesàro averaged functionals are typically applicable to Harris recurrent Markov chains (discrete time) and processes (continuous time). See Sigman [20] and Glynn and Sigman [8]. We believe that similar results for MPP's on $\mathbb{R}$ which are not necessarily ergodic, are of mathematical interest and will turn out to be useful for future applications. Essentially, the results of Section 7 are mathematical applications of these results.

Some of the uniform Cesàro limit results in this research, especially those concerning total variation convergence, can also be obtained by shift-coupling methods. See for this approach Thorisson [22]. We believe that the direct proofs of the present paper are of interest in themselves.

Some final remarks. We will sometimes write $\mathbb{E}X^2$ and $\mathbb{E}XY$ (for random variables

X and Y) when $\mathbb{E}(X^2)$ and $\mathbb{E}(XY)$ is meant. If an event A is described by a complicated expression, we will write $\mathbf{1}A$ for the indicator function 1_A. When talking about Radon-Nikodym derivatives, the attribute a.s. (almost surely) is sometimes suppressed. We will often make use of the time parameters t, n, k, i, and j. The first is a continuous-time parameter, the others are discrete-time parameters.

2 PRELIMINARIES

We formalize some of the notions already mentioned in Section 1 and give some other definitions and notations. The relationships between time-stationary distribution and Palm distribution, and between Palm distribution and modified Palm distribution are characterized in terms of Radon-Nikodym densities. These densities represent the main tool to accomplish the transitions between the many distributions studied in this research.

In the following $\mathbb{R}, \mathbb{R}^+, \mathbb{Z}$ and $\mathbb{N}_0$ are the set of reals, the set of nonnegative reals, the set of integers, and the set of nonnegative integers. K is a metric space, assumed to be complete and separable. $\operatorname{Bor}\mathbb{R}$ and $\operatorname{Bor}K$ denote the Borel σ-fields on $\mathbb{R}$ and K. A *marked point process on* $\mathbb{R}$ *with mark space* K is a random element Φ in the set of all integer-valued measures φ on the σ-field $\operatorname{Bor}\mathbb{R} \times \operatorname{Bor}K$ such that:

$$\varphi(A \times K) < \infty \text{ for all bounded } A \in \operatorname{Bor}\mathbb{R}.$$

(So, an MPP is a random counting measure on $\mathbb{R} \times K$.) Let M_K be this set and endow it with the natural σ-field $\mathcal{M}_K$ (generated by the sets $[\varphi(A \times L) = k] := \{\varphi \in M_K : \varphi(A \times L) = k\}$, $k \in \mathbb{N}_0$, $L \in \operatorname{Bor}K$ and $A \in \operatorname{Bor}\mathbb{R}$). The distribution of Φ will be denoted by P, a probability measure on $(M_K, \mathcal{M}_K)$. The atoms of $\varphi \in M_K$ are denoted by $(X_i(\varphi), k_i(\varphi))$, $i \in \mathbb{Z}$, with the convention that

$$\ldots \leq X_{-1}(\varphi) \leq X_0(\varphi) \leq 0 < X_1(\varphi) \leq X_2(\varphi) \leq \ldots$$

$X_i(\varphi)$ is interpreted as the i'th *occurrence* (*event, point*) of φ, $k_i(\varphi)$ as the accessory *mark*. For a subset L of marks we write $X_i^L(\varphi)$, $i \in \mathbb{Z}$, for the "i'th L-point of φ". For $L \in \operatorname{Bor}K$ we are only interested in counting measures $\varphi \in M_K$ which do not have multiple occurrences (so, the inequalities in the above convention are strict) and which have infinitely many L-points on both half lines $(-\infty, 0]$ and $(0, \infty)$. Denote this set of φ's by M_L^∞, the subset with $X_0^L(\varphi) = 0$ by M_L^0, and the corresponding σ-fields by

$\mathcal{M}_L^\infty = M_L^\infty \cap \mathcal{M}_K$ and $\mathcal{M}_L^0 = M_L^0 \cap \mathcal{M}_K$. The relation between $\varphi \in M_K^\infty$ and the set $\{(X_i(\varphi), k_i(\varphi)) : i \in \mathbb{Z}\}$ of its atoms is also described by

$$\begin{aligned}\varphi(A \times L) &= \#\{i \in \mathbb{Z} : X_i(\varphi) \in A \text{ and } k_i(\varphi) \in L\} \\ &= \#\{i \in \mathbb{Z} : X_i^L(\varphi) \in A\},\end{aligned}$$

$A \in \operatorname{Bor}\mathbb{R}$ and $L \in \operatorname{Bor} K$. Note also that $\varphi(\{0\} \times L) = 1$ if φ is an element of M_L^0. For $\varphi \in M_L^\infty$ we write $\alpha_i^L(\varphi) := X_{i+1}^L(\varphi) - X_i^L(\varphi)$, the i'th *interval length* between the L-points $X_i^L(\varphi)$ and $X_{i+1}^L(\varphi)$. For a realization $\varphi \in M_K^\infty$ and a time point $t \in \mathbb{R}$ the element $T_t\varphi = \varphi(t + \cdot)$ of M_K^∞ arises from φ by shifting the origin to t and considering the realization from this new position. So, $T_t\varphi$ can be represented by the set $\{(X_j(\varphi) - t, k_j(\varphi)) : j \in \mathbb{Z}\}$ containing its atoms. The corresponding MPP is denoted by $T_t\Phi = \Phi(t + \cdot)$.

We will assume that the MPP Φ has the same distribution seen from all time points $t \in \mathbb{R}$. Formally this means that $\Phi(t + \cdot)$ has the same distribution as Φ for all $t \in \mathbb{R}$, i.e. that Φ is *time-stationary*. It will also be assumed that w.p.1 Φ has infinitely many points on both half lines, all of them not multiple. That is, $P(M_K^\infty) = 1$. Furthermore, with $\lambda(L) := \mathbb{E}\Phi((0,1] \times L)$ the *intensity of the L-points*, it is assumed that the *intensity* $\lambda(K)$ is finite. We will only consider subsets L of marks such that w.p.1 Φ has infinitely many L-points on both half lines, all not multiple. So, talking about $L \in \operatorname{Bor} K$ we implicitly assume that $P(M_L^\infty) = 1$.

Two types of shifts will be considered. The *time shifts* $T_t : M_K^\infty \to M_K^\infty$, $t \in \mathbb{R}$, are defined above. Note that $PT_t^{-1} = P$ for all $t \in \mathbb{R}$, by time-stationarity. For fixed $L \in \operatorname{Bor} K$ the *point shift* $\theta_{n,L} : M_L^\infty \to M_L^\infty$, $n \in \mathbb{Z}$, moves the origin to the n'th L-point. It is defined by $\theta_{n,L}\varphi := \varphi(X_n^L(\varphi) + \cdot)$, also represented by the set of its atoms, i.e. by $\{(X_j(\varphi) - X_n^L(\varphi), k_j(\varphi)) : j \in \mathbb{Z}\}$. Note that shifting the origin to the i'th L-point, followed by another shift of the origin to the j'th L-point seen from the last position, is the same as shifting the origin in one step to the $(i + j)$'th L-point. That is, $\theta_{j,L} \circ \theta_{i,L} = \theta_{i+j,L}$ for all $i, j \in \mathbb{Z}$. The probability measure $P_{n,L} := P\theta_{n,L}^{-1}$, $n \in \mathbb{Z}$, on $(M_L^\infty, \mathcal{M}_L^\infty)$ arises from P by shifting the origin to the n'th L-point. To illustrate our notation we point out that $[\theta_{n,L}\varphi \in B] = \{\varphi \in M_L^\infty : \theta_{n,L}\varphi \in B\}$ and $[T_t\varphi \in A] = \{\varphi \in M_K^\infty : T_t\varphi \in A\}$, $B \in \mathcal{M}_L^\infty$, $A \in \mathcal{M}_K^\infty$, $n \in \mathbb{Z}$ and $t \in \mathbb{R}$.

For $L \in \operatorname{Bor} K$ the *Palm distribution* P_L^0 of Φ (or rather P) *with respect to* L is defined by

$$P_L^0(A) := \frac{1}{\lambda(L)} \mathbb{E}\left[\sum_{i=1}^{\Phi((0,1]\times L)} 1_A(\theta_{i,L}\Phi)\right], \qquad A \in \mathcal{M}_L^\infty. \tag{1}$$

Note that under P_L^0 there is an occurrence in the origin with mark in L, i.e. $P_L^0(M_L^0) = 1$, and that working with P_L^0 does not mean that we restrict ourselves to the L-points. Intuitively, P_L^0 is the conditional distribution of Φ given the occurrence of an L-point in the origin. This interpretation follows immediately from the generalization to marked PP's of the local characterization theorem. See, e.g., Theorem 10 in Nieuwenhuis [18]. Mind the difference between P_L^0 and $P_{0,L}$, in notation as well as in interpretation. Several probability measures on $(M_L^\infty, \mathcal{M}_L^\infty)$ have been defined so far: P, P_L^0, $P_{n,L}$. In this research expectations with respect to these measures are denoted by E, E_L^0, $E_{n,L}$, respectively. When another probability measure Q on $(M_L^\infty, \mathcal{M}_L^\infty)$ is considered, we will write E_Q for the corresponding expectation. Expectation with respect to a universal probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is (as in (2.1)) denoted by $\mathbb{E}$. The probability measure P_L^0 has the following properties:

$$P_L^0\theta_{n,L}^{-1} = P_L^0 \qquad \text{for all } n \in \mathbb{Z}, \tag{2}$$

$$P(A) = \lambda(L) \int_0^\infty P_L^0[X_1^L(\varphi) > u;\ \varphi(u + \cdot) \in A]du, \qquad A \in \mathcal{M}_L^\infty. \tag{3}$$

See Franken et al. [7], Matthes, Kerstan and Mecke [12], Kallenberg [9], and Brandt, Franken and Lisek [4] for more information and for proofs. Relation (2.2) means that an MPP Φ_L^0 with distribution P_L^0 (a so-called Palm L-process) has the same distribution seen from all L-points. We call this *L-event-stationarity.* With the choice $A = M_L^\infty$ in (2.3) we obtain $E_L^0\alpha_0^L = 1/\lambda(L)$.

The *inversion formula* (2.3) expresses P in terms of P_L^0; the definition in (2.1) expresses P_L^0 in terms of P. There is another way of going from P_L^0 to P (and vice versa). The essence of the approach is contained in the next theorem. It is proved in Nieuwenhuis [17]; the extension to **marked** point processes is straightforward.

(We need some definitions first. Two probability measures Q_1 and Q_2 on a common measurable space are called *equivalent* (notation $Q_1 \sim Q_2$) if they have the same null-sets. Q_1 is *dominated by* Q_2 (notation $Q_1 \ll Q_2$) if all Q_2-null-sets are also Q_1-null-sets; a Radon-Nikodym density is denoted by dQ_1/dQ_2.)

theorem 2.1 *For $n \in \mathbb{Z}$ and $L \in \operatorname{Bor} K$ we have:*

(i) $P_{n,L} \sim P_L^0$,

(ii) $\dfrac{dP_{n,L}}{dP_L^0} = \lambda(L)\alpha_{-n}^L \quad P_L^0$-a.s.

Suppose that $f : M_L^0 \to \mathbb{R}$ is P_L^0-integrable. Since $E_L^0 f = E_{0,L}(f/\alpha_0^L)/\lambda(L)$ by part (ii) with $n = 0$, we obtain:

$$E_L^0 f = \frac{1}{\lambda(L)} E\left(\frac{1}{\alpha_0^L} f \circ \theta_{0,L}\right). \tag{4}$$

This relation expresses a transition from P to P_L^0 where $P_{0,L}$ is used as a bridge. At first the origin is shifted to the last L-point on its left, to X_0^L. Then the importance of the realizations is changed by way of the weight function $(\lambda(L)\alpha_0^L)^{-1}$. Similarly, a transition from P_L^0 to P can be effected by first changing the importance of the realisations by the weight function $\lambda(L)\alpha_0^L$, followed by shifting the origin to a time point which is chosen at random in $[X_0^L, X_1^L)$. See Sections 1 and 2 of Nieuwenhuis [18] for more information about two-step transitions of this type. Similarly, if $g : M_L^\infty \to \mathbb{R}$ is P-integrable with $Eg = Eg \circ \theta_{0,L}$, then the P-expectation of g can be transformed into a P_L^0-expectation:

$$Eg = E_{0,L} g = \lambda(L) E_L^0(\alpha_0^L g). \tag{5}$$

Relations (2.4) and (2.5) are very useful, see Nieuwenhuis [16], [17] and [18], and Thorisson [22]. Their profit can be demonstrated by mentioning an immediate consequence: $E(1/\alpha_0^L) = \lambda(L) = 1/E_L^0(\alpha_0^L)$. For more applications of Theorem 2.1 we refer to Nieuwenhuis [18]. The approach in (2.4) and (2.5), where $P_{0,L}$ is used as a bridge between P_L^0 and P, is very common in the present research.

Consider the following invariant σ-fields:

$$\begin{aligned} \mathcal{I}'_L &:= \{A \in \mathcal{M}_L^\infty : T_t^{-1}A = A \text{ for all } t \in \mathbb{R}\} \text{ and} \\ \mathcal{I}_L &:= \{A \in \mathcal{M}_L^\infty : \theta_{1,L}^{-1}A = A\}. \end{aligned} \tag{6}$$

So, $\mathcal{I}'_L$ contains the sets which are invariant under all time shifts. It is not hard to prove that $\mathcal{I}_L$ contains all sets invariant under all point shifts $\theta_{n,L},\ \ n \in \mathbb{Z}$. See also Lemma 3.1(a). Since $\alpha_i^L \circ \theta_{k,L} = \alpha_{i+k}^L$, it follows immediately from L-event-stationarity that under P_L^0 the random sequences $(\alpha_{i+k}^L)_{i\in\mathbb{Z}}$ and $(\alpha_i^L)_{i\in\mathbb{Z}}$ have the same distribution for all $k \in \mathbb{Z}$. We say that $(\alpha_i^L)_{i\in\mathbb{Z}}$ is P_L^0-*stationary.* By Birkhoff's ergodic theorem (or law of large numbers) we obtain that $n^{-1}\sum_{i=1}^{n}\alpha_i^L$ tends (as $n\to\infty$) to a limit which is possibly random and can be characterized as a conditional expectation on the σ-field $\mathcal{I}_L$. That is,

$$\frac{1}{n}\sum_{i=1}^{n}\alpha_i^L \to \bar{\alpha}_0^L := E_L^0(\alpha_0^L|\mathcal{I}_L) \quad P_L^0\text{-a.s.} \tag{7}$$

From Theorem 3 in Nieuwenhuis [18] it follows that this convergence also holds P-a.s. The MPP Φ (and its distribution P) is called *pseudo-L-ergodic* if the limit in (2.7) is nonrandom, i.e. if

$$\bar{\alpha}_0^L = \frac{1}{\lambda(L)} \quad P_L^0\text{-a.s.} \tag{8}$$

P (and Φ) is *ergodic* if $P(A)\in\{0,1\}$ for all $A\in\mathcal{I}'_K$, and P_L^0 is *ergodic* if $P_L^0(A)\in\{0,1\}$ for all $A\in\mathcal{I}_L$. In Section 3 an example will be given of a nonergodic MPP which is pseudo-L-ergodic.

We introduce more probability measures. Let Q_L^0 on $(M_L^\infty,\mathcal{M}_L^\infty)$ be defined by

$$Q_L^0(B) := E(E_L^0(1_B|\mathcal{I}_L)), \qquad B\in\mathcal{M}_L^\infty. \tag{9}$$

Since $E_L^0(1_B|\mathcal{I}_L) = P_L^0(B|\mathcal{I}_L)$ equals $\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n}\mathbf{1}[\theta_{i,L}\varphi\in B]$, we can interpret $Q_L^0(B)$ as the expected value (under P) of the long-run proportion of the L-points where "B is seen". This formal probability measure is called the *modified Palm distribution of P with respect to L*. It seems to be more in accordance with the intuitive definition (1.2) of P_L^0 than P_L^0 itself. This is expressed in the following theorem. In this result Q_L^0 is approximated when starting from P. For unmarked point processes it is proved in Section 4 of Nieuwenhuis [18]; the generalization to MPP's is straightforward.

theorem 2.2 *Let $L\in \mathrm{Bor}\,K$. Then Q_L^0 is equivalent to P_L^0 and*

$$\frac{dQ_L^0}{dP_L^0} = \lambda(L)\bar{\alpha}_0^L \quad P_L^0\text{-}a.s. \tag{10}$$

Q_L^0 and P_L^0 are equal iff Φ is pseudo-L-ergodic. The supremum

$$\sup_{B\in\mathcal{M}_L^\infty}\left|\frac{1}{n}\sum_{i=1}^{n}P[\theta_{i,L}\varphi\in B]-Q_L^0(B)\right| = \frac{\lambda(L)}{2}E_L^0\left|\frac{1}{n}\sum_{i=1}^{n}\alpha_{-i}^L-\bar{\alpha}_0^L\right|$$

tends to 0 *as* $n \to \infty$.

As a consequence we can conclude now that the obvious formalization of the intuitive definition (1.2) of P_L^0 is only correct if Φ is pseudo-L-ergodic. If this weak ergodicity condition is not satisfied, then it is the modified Palm distribution Q_L^0 and not the ordinary Palm distribution P_L^0 which fits this intuition. Recall that $\alpha_i^L = \alpha_0^L \circ \theta_{i,L}$. In view of (1.2) one might at first sight expect that

$$\frac{1}{n}\sum_{i=1}^{n} E\alpha_i^L \to E_L^0\alpha_0^L = \frac{1}{\lambda(L)}. \tag{11}$$

However, since the limit result in (2.7) also holds P-a.s., $n^{-1}\sum_{i=1}^{n} E\alpha_i^L$ will (under weak additional conditions) tend to $E\bar{\alpha}_0^L$. By (2.5) and conditioning on $\mathcal{I}_L$ we have:

$$E\bar{\alpha}_0^L = \lambda(L)E_L^0\left(\alpha_0^L\bar{\alpha}_0^L\right) = \lambda(L)E_L^0\left[\left(\bar{\alpha}_0^L\right)^2\right] \geq \lambda(L)\left(E_L^0\bar{\alpha}_0^L\right)^2 = \frac{1}{\lambda(L)}.$$

Equality holds iff $\bar{\alpha}_0^L = 1/\lambda(L) \quad P_L^0$-a.s., i.e. iff Φ is pseudo-L-ergodic. So, the intuitive limit in (2.11) is not necessarily correct. Note, however, that by (2.5) and (2.10) $E\bar{\alpha}_0^L = E_{Q_L^0}\alpha_0^L$. All these arguments make Theorem 2.2 less surprising.

Modified Palm distributions have further important properties. By replacing B in the convergence part of Theorem 2.2 by $[\theta_{k,L}\varphi \in B]$ for fixed $k \in \mathbb{Z}$, it follows inmmediately that $Q_L^0\theta_{k,L}^{-1} = Q_L^0$. Just like P_L^0, the modified Palm distribution Q_L^0 is event-L-stationary. Note that the Radon-Nikodym derivative in (2.10) is $\mathcal{I}_L$-measurable. From this observation it follows that, although $Q_L^0(A)$ and $P_L^0(A)$ may differ, the conditional probabilities $Q_L^0(A|\mathcal{I}_L)$ and $P_L^0(A|\mathcal{I}_L)$ are equal:

$$Q_L^0(A|\mathcal{I}_L) = P_L^0(A|\mathcal{I}_L) \quad \text{a.s.} \tag{12}$$

under both Q_L^0 and P_L^0.

A family $(Y_t)_{t\in I}$ of integrable random variables is called *uniformly integrable* if $\sup_{t\in I}\mathbb{E}\left(|Y_t|\mathbf{1}\left[|Y_t| \geq b\right]\right) \to 0$ as $b \to \infty$. For a probability measure Q we will abbreviate "uniformly Q-integrable" to "u.i. under Q". The following lemma will be applied in Sections 4, 5, and 6. It follows immediately from Theorem 5.4 in Billingsley [3]. The notation $\xrightarrow{d}$ expresses convergence in distribution.

lemma 2.1 *Let $Y, Y_1, Y_2, \ldots$ be nonnegative, real-valued r.v.'s with $Y_n \xrightarrow{d} Y$. Then $(Y_n)_{n\geq 1}$ is uniformly integrable if and only if*

$$\mathbb{E}Y < \infty, \quad \mathbb{E}Y_n < \infty \text{ for all } n \in \mathbb{N}, \quad \text{and } \mathbb{E}Y_n \to \mathbb{E}Y.$$

Let Q_1 and Q_2 be probability measures on a common measurable space, both dominated by a σ-finite measure μ and having densities h_1 and h_2 respectively. The *total variation distance* between Q_1 and Q_2 is defined by

$$d(Q_1, Q_2) := \int |h_1 - h_2| d\mu. \tag{13}$$

It can be proved that

$$d(Q_1, Q_2) = 2 \sup_A |Q_1(A) - Q_2(A)| = 2(Q_1[h_1 \geq h_2] - Q_2[h_1 \geq h_2]). \tag{14}$$

(Equality of the left-hand part and the right-hand part follows immediately by writing the integral in (2.13) as the summation of the integrals over $[h_1 \geq h_2]$ and $[h_1 < h_2]$. For the second equality we note that for A such that $Q_1(A) \geq Q_2(A)$ we have:

$$\begin{aligned} Q_1(A) - Q_2(A) &\leq Q_1(A \cap [h_1 \geq h_2]) - Q_2(A \cap [h_1 \geq h_2]) \\ &\leq Q_1[h_1 \geq h_2] - Q_2[h_1 \geq h_2]. \end{aligned}$$

For other A, consider the complements.)

3 CONDITIONING ON INVARIANT σ-FIELDS

At first sight each family of shifts induces its own type of invariant sets. Fortunately, the corresponding invariant σ-fields coincide. The corollaries of this result for ergodicity conditions are studied. It also has its influence on the relationship between time-stationary and event-stationary distributions. Especially, several conditional expectations (i.e, random limits in laws of large numbers) are compared. The rather technical results of this section will be applied several times in Sections 4 to 6.

Recall the definitions of $\mathcal{I}_L$ and $\mathcal{I}'_L$ in (2.6). The following lemma is a straightforward generalization of Lemma 2 in Nieuwenhuis [18].

lemma 3.1 *Let* $L \in \operatorname{Bor} K$. *Then:*
(a) *If* $A \in \mathcal{I}_L$, *then* $\theta_{i,L}^{-1} A = A$ *for all* $i \in \mathbb{Z}$.
(b) $\mathcal{I}_L = \mathcal{I}'_L$.

Note that as a consequence of Lemma 3.1 every $\mathcal{I}_L$-measurable function $f : M_L^\infty \to [0, \infty)$ satisfies

$$f \circ \theta_{i,L}(\varphi) = f(\varphi) \qquad \text{and} \qquad f \circ T_t(\varphi) = f(\varphi) \tag{1}$$

for all $\varphi \in M_L^\infty$, $i \in \mathbb{Z}$, and $t \in \mathbb{R}$. We will often use this result, sometimes without mention. For instance, $[\theta_{i,L}\varphi \in B] = B$ for all $B \in \mathcal{I}_L$. By the second part of Theorem 2.2 it follows that Q_L^0 and P coincide on the invariant σ-field $\mathcal{I}_L$, while by the first part of this theorem P_L^0 and P only agree for null-sets from $\mathcal{I}_L$. Especially, (2.12) is also valid under P.

In view of Section 6 we next consider two nonempty sets of marks. Let L, $L' \in \operatorname{Bor} K$. Furthermore, set

$$\begin{aligned}
M_{L,L'}^\infty &:= M_L^\infty \cap M_{L'}^\infty \text{ and } \mathcal{M}_{L,L'}^\infty := M_{L,L'}^\infty \cap \mathcal{M}_K, \\
\mathcal{I}_{L,L'} &:= \{A \in \mathcal{M}_{L,L'}^\infty : \theta_{1,L}^{-1} A = A\}, \\
\mathcal{I}'_{L,L'} &:= \{A \in \mathcal{M}_{L,L'}^\infty : T_t^{-1} A = A \text{ for all } t \in \mathbb{R}\}.
\end{aligned}$$

So, every $\varphi \in M_{L,L'}^\infty$ has infinitely many points of both types L and L' on both half lines. In the presence of two sets of marks, L and L', the mappings $\theta_{i,L}, \theta_{i,L'}$, and T_t will always be restricted to $M_{L,L'}^\infty$. The following relations can easily be proved:

$$\begin{aligned}
&\mathcal{I}'_L \cap M_{L'}^\infty = \mathcal{I}'_{L,L'} \quad \text{and} \quad \mathcal{I}_L \cap M_{L'}^\infty = \mathcal{I}_{L,L'}; \\
&\mathcal{I}'_{L,L'} \subset \mathcal{I}'_L \qquad\qquad \text{and} \quad \mathcal{I}_{L,L'} \subset \mathcal{I}_L.
\end{aligned} \tag{2}$$

The second equality in part (b) of the next lemma states that the invariant σ-fields induced by the families $\{\theta_{n,L} : n \in \mathbb{Z}\}$ and $\{\theta_{n,L'} : n \in \mathbb{Z}\}$ of point shifts coincide if the shifts are restricted to $M_{L,L'}^\infty$.

lemma 3.2 *Let $L, L' \in \mathrm{Bor}\, K$. Then:*
(a) *If* $A \in \mathcal{I}_{L,L'}$, *then* $\theta_{i,L}^{-1} A = A$ *for all* $i \in \mathbb{Z}$;
(b) $\mathcal{I}'_{L,L'} = \mathcal{I}_{L,L'} = \mathcal{I}_{L',L}$.

Proof. Since $\mathcal{I}_{L,L'} \subset \mathcal{I}_L$, part (a) follows from Lemma 3.1(a). Part (b) is an immediate consequence of Lemma 3.1(b) and (3.2) since

$$\begin{aligned}
\mathcal{I}_{L,L'} &= \mathcal{I}_L \cap M_{L'}^{\infty} = \mathcal{I}'_L \cap M_{L'}^{\infty} = \mathcal{I}'_{L,L'} = \mathcal{I}'_{L',L} \\
&= \mathcal{I}'_{L'} \cap M_L^{\infty} = \mathcal{I}_{L'} \cap M_L^{\infty} = \mathcal{I}_{L',L}.
\end{aligned}$$ □

As a consequence every $\mathcal{I}_{L,L'}$-measurable function $f{:}M_{L,L'}^{\infty} \to [0,\infty)$ satisfies

$$f \circ \theta_{i,L}(\varphi) = f(\varphi), \quad f \circ \theta_{i,L'}(\varphi) = f(\varphi), \quad \text{and } f \circ T_t(\varphi) = f(\varphi) \tag{3}$$

for all $\varphi \in M_{L,L'}^{\infty}$, $i \in \mathbb{Z}$, and $t \in \mathbb{R}$.

Next a time-stationary MPP Φ with distribution P is put upon the stage. Since $\mathcal{I}'_L \subset \mathcal{I}'_K$ and $\mathcal{I}'_L = \mathcal{I}'_K \cap \mathcal{M}_L^{\infty}$, the σ-field $\mathcal{I}'_K$ in the definition of ergodicity of P in Section 2 may equivalently be replaced by $\mathcal{I}'_L$. As a consequence of Lemma 3.1(b) we obtain:

$$\begin{array}{llcll}
P & \text{is ergodic} & \Longleftrightarrow & P_L^0 & \text{is ergodic,} \\
P & \text{is ergodic} & \Longrightarrow & P & \text{is pseudo-}L\text{-ergodic.}
\end{array} \tag{4}$$

If P is pseudo-L-ergodic, then it is not necessarily ergodic. Pseudo-L-ergodicity does not necessarily imply pseudo-L'-ergodicity. See the example at the end of this section.

For $t \geq 0$ the random variable $N_L(t) : M_L^{\infty} \to \mathbb{N}_0$ is defined by $N_L(t,\varphi) := \varphi((0,t] \times L)$ and can be read as the number of L-points in the interval $(0,t]$. The time-stationary mean value of $1/\alpha_0^L$, the reciprocal of the length of the interval in which the origin is situated, is equal to the time-stationary mean number of L-points per unit of time. This in turn is equal to the reciprocal of the L-event-stationary mean value of α_0^L. See the equalities following (2.5). A generalization of these results in terms of conditional expectations is formulated below. To have a good comparison, we will write $E_L^0(\alpha_0^L|\mathcal{I}_L)$ in full instead of $\bar{\alpha}_0^L$.

lemma 3.3 *Let L, $L' \in \operatorname{Bor} K$ be nonempty. The following relations hold P-a.s. as well as P^0_L-a.s.*

(a) $E(\frac{1}{\alpha^L_0}|\mathcal{I}_L) = E(N_L(1)|\mathcal{I}_L)$,
(b) $E^0_L(\alpha^L_0|\mathcal{I}_L) > 0$,
(c) $E(\frac{1}{\alpha^L_0}|\mathcal{I}_L) = \frac{1}{E^0_L(\alpha^L_0|\mathcal{I}_L)}$.

Parts (a), (b), *and* (c) *remain valid if $\mathcal{I}_L$ is replaced by $\mathcal{I}_{L,L'}$. The resulting relations hold $P^0_{L'}$-a.s. as well.*

Proof. Let $A \in \mathcal{I}_L$. Note that $\alpha^L_0 = \alpha^L_0 \circ \theta_{0,L}$. By (3.1), (2.5), and (2.1) we have

$$E(1_A E(\frac{1}{\alpha^L_0}|\mathcal{I}_L)) = E(1_A \frac{1}{\alpha^L_0}) = E_{0,L}(1_A \frac{1}{\alpha^L_0}) = \lambda(L) P^0_L(A) = E(1_A N_L(1)).$$

So, part (a) holds P-a.s., $P_{0,L}$-a.s., and hence P^0_L-a.s. Set $B := [E^0_L(\alpha^L_0|\mathcal{I}_L) \leq 0]$. Then

$$0 \geq E^0_L(1_B E^0_L(\alpha^L_0|\mathcal{I}_L)) = E^0_L(1_B \alpha^L_0).$$

Since $P^0_L[\alpha^L_0 > 0] = 1$, we obtain for the complement B^c of B:

$$P^0_L(B^c) = 1 \qquad \text{and} \qquad P(B^c) = E(1_{B^c} \circ \theta_{0,L}) = P_{0,L}(B^c) = 1.$$

Part (b) follows. Let again $A \in \mathcal{I}_L$. By (3.1) and (2.5) we have

$$\begin{aligned}
E\left(1_A \frac{1}{E^0_L(\alpha^L_0|\mathcal{I}_L)}\right) &= E\left(1_A \circ \theta_{0,L} \frac{1}{E^0_L(\alpha^L_0|\mathcal{I}_L) \circ \theta_{0,L}}\right) \\
&= \lambda(L) E^0_L\left(\alpha^L_0 1_A \frac{1}{E^0_L(\alpha^L_0|\mathcal{I}_L)}\right) \\
&= \lambda(L) P^0_L(A) = E\left(1_A \frac{1}{\alpha^L_0}\right) = E\left(1_A E\left(\frac{1}{\alpha^L_0}|\mathcal{I}_L\right)\right).
\end{aligned}$$

(In the third equality we conditioned on $\mathcal{I}_L$.) Consequently, part (c) holds P-a.s., and by (3.1) also P^0_L-a.s. Since $\mathcal{I}_{L,L'} = \mathcal{I}_L \cap M^\infty_{L'}$ and $P(M^\infty_{L,L'}) = 1$ by assumption (see Section 2), it is obvious that (a), (b) and (c) remain valid if $\mathcal{I}_L$ is replaced by $\mathcal{I}_{L,L'}$. By (3.3) the resulting expressions also hold under $P_{0,L'}$, and hence under $P^0_{L'}$. $\square$

In view of Section 6 we need another lemma for the case that two nonempty sets $L, L' \in \operatorname{Bor} K$ are involved. For $i \in \mathbb{Z}$ the random variable $\xi_i : M^\infty_{L,L'} \to [0,\infty)$ is defined by

$$\xi_i(\varphi) := \varphi((X^L_i(\varphi), X^L_{i+1}(\varphi)] \times L'), \quad \varphi \in M^\infty_{L,L'}. \tag{5}$$

So, $\xi_i(\varphi)$ is the number of L'-points in the interval $(X^L_i(\varphi), X^L_{i+1}(\varphi)]$. Note that $\xi_i(\theta_{1,L}\varphi) = \xi_{i+1}(\varphi)$ for all $\varphi \in M^\infty_{L,L'}$. Hence, the random sequence (ξ_j) is P^0_L-stationary. In the ergodic case it is well-known that the (long-run) average number of L'-arrivals between two successive L-arrivals is equal to the ratio $\lambda(L')/\lambda(L)$ of the average numbers of L'- and L-arrivals per unit of time; see, e.g., Relation (3.4.2) in Baccelli and Brémaud [2]. A similar result holds in the non-ergodic case, the case in which long-run averages are possibly nondegenerate random variables characterized as conditional expectations on the invariant σ-field. Recall the definition of $N_L(t)$ preceding Lemma 3.3, and note that $E(N_L(1)|\mathcal{I}_{L,L'}) > 0$ P-a.s. since (by (2.1)) the set $B := [E(N_L(1)|\mathcal{I}_{L,L'}) \le 0]$ satisfies

$$0 \ge E\left(1_B E(N_L(1)|\mathcal{I}_{L,L'})\right) = E(1_B N_L(1)) = \lambda(L) P^0_L(B).$$

lemma 3.4 *Let $L, L' \in \operatorname{Bor} K$ be nonempty. Then*

$$E^0_L\left(\xi_0|\mathcal{I}_{L,L'}\right) = \frac{E\left(N_{L'}(1)|\mathcal{I}_{L,L'}\right)}{E(N_L(1)|\mathcal{I}_{L,L'})} = \frac{E^0_L\left(\alpha^L_0|\mathcal{I}_{L,L'}\right)}{E^0_{L'}(\alpha^{L'}_0|\mathcal{I}_{L,L'})} \qquad P^0_L\text{-},\ \ P^0_{L'}\text{-}, \quad \text{and} \quad P\text{-a.s..}$$

Proof. If $t_1, t_2 \ge 0$ with $t_1 \le t_2$, we write $N_{L'}(t_1, t_2] := N_{L'}(t_2) - N_{L'}(t_1)$. Note that, with this notation, $\xi_i = N_{L'}(X^L_i, X^L_{i+1}]$ if $X^L_i \ge 0$. Since (ξ_i) is P^0_L-stationary, we obtain

$$\frac{1}{n} N_{L'}(0, X^L_n] \to E^0_L(\xi_0|\mathcal{I}_{L,L'}) \qquad P^0_L\text{-a.s.} \tag{6}$$

(Note that $N_{L'}(0, X^L_n] = \sum_{i=0}^{n-1} \xi_i \quad P^0_L$-a.s., and apply Birkhoff's ergodic theorem.) Since $\xi_i = \xi_i \circ \theta_{0,L}$, Relation (3.6) holds P-a.s. as well; cf. Theorem 2.1 (i). As

$$\frac{N_L(t)}{t} \frac{N_{L'}(0, X^L_{N_L(t)}]}{N_L(t)} \le \frac{1}{t} N_{L'}(0,t] \le \frac{N_{L'}(0, X^L_{N_L(t)+1}]}{N_L(t)+1} \frac{N_L(t)+1}{t}$$

on $[N_L(t) > 0]$, and

$$\frac{N_L(t)}{t} \to E(N_L(1)|\mathcal{I}_{L,L'}) \quad \text{and} \quad N_L(t) \to \infty \ \ P\text{-a.s.,} \tag{7}$$

we obtain

$$\frac{N_{L'}(t)}{t} \to E(N_L(1)|\mathcal{I}_{L,L'})E^0_L(\xi_0|\mathcal{I}_{L,L'}) \ \ P\text{-a.s.} \tag{8}$$

Replacing L by L' in (3.7) yields $E(N_{L'}(1)|\mathcal{I}_{L,L'})$ as another limit of $t^{-1}N_{L'}(t)$, P-a.s. Hence,

$$E^0_L(\xi_0|\mathcal{I}_{L,L'}) = \frac{E\left(N_{L'}(1)|\mathcal{I}_{L,L'}\right)}{E(N_L(1)|\mathcal{I}_{L,L'})} \quad P\text{-a.s.} \tag{9}$$

By (3.3), Relation (3.9) holds under $P_{0,L}$ and under $P_{0,L'}$ as well. By Theorem 2.1 it also holds with P^0_L or $P^0_{L'}$ instead of P. Lemma 3.3 yields

$$E(N_L(1)|\mathcal{I}_{L,L'}) = \frac{1}{E^0_L\left(\alpha^L_0|\mathcal{I}_{L,L'}\right)} \quad \text{and} \quad E(N_{L'}(1)|\mathcal{I}_{L,L'}) = \frac{1}{E^0_{L'}(\alpha^{L'}_0|\mathcal{I}_{L,L'})}$$

P^0_L-, $P^0_{L'}$-, and P-a.s. Combining the above observations completes the proof. □

$P^0_{L'}$-expectations can directly be expressed in terms of P^0_L-expectations by Neveu's *exchange formula* (or *cycle formula*)

$$E^0_{L'}f = \frac{\lambda(L)}{\lambda(L')}E^0_L\left[\sum_{i=1}^{\xi_0} f\circ\theta_{i,L'}\right], \tag{10}$$

where $f : M^\infty_{L,L'} \to \mathbb{R}$ is $P^0_{L'}$-integrable. This can be proved by replacing 1_A in (2.1) by $\sum_{i=1}^{\xi_0} f\circ\theta_{i,L'}$; see also Neveu [15].

Non-trivial mixtures of time-stationary distributions are non-ergodic. Mixtures of two such distributions not only characterize the non-ergodic case (see, e.g., (7.2.3) in Baccelli & Brémaud [2]), but also give a good and simple illustration of the above results, especially of those concerning pseudo-ergodicity.

Example 3.1. In an insurance company claims of three types (say L_1, L_2 and L_3) are planned to come in according to one of two possible scenarios. In scenario 1 the pairs of claim epochs and accompanying claim types follow a time-stationary MPP with distribution P_1 and intensities $\lambda_1(L_1) = 3, \lambda_1(L_2) = 2$ and $\lambda_1(L_3) = 1$ for the three types; this scenario occurs with probability $q \in (0, 1)$. The second scenario, occurring with probability $1-q$, describes these pairs by a time-stationary MPP with distribution P_2 and with intensities $\lambda_2(L_1) = 1, \lambda_2(L_2) = 2$ and $\lambda_2(L_3) = 3$. It is assumed that under both scenarios the MPP Φ is pseudo-ergodic with respect to each of the types. So, the (long-run) average interval length between successive claims of type L_i is a constant, $1/\lambda_j(L_i)$ for scenario j.

The distribution $P = qP_1 + (1-q)P_2$ conducts the series of claim epochs and claim types irrespective of the scenario. It is the distribution of the resulting MPP. Obviously, this Φ is time-stationary but not ergodic; the set $[n^{-1}\sum_{k=1}^{n} \alpha_k^{L_1} \to 1/3]$ is an element of $\mathcal{I}_{L_1}$ and has P-probability $q \notin \{0, 1\}$. But how about pseudo-ergodicity? Denoting the long-run average interval length between successive claims of type L_i, i.e. $E^0_{L_i}(\alpha_0^{L_i}|\mathcal{I}_{L_i})$, by $\bar{\alpha}_0^{L_i}$, it is an easy excercise to prove that

$$P[\bar{\alpha}_0^{L_1} = \frac{1}{3}] = q \quad \text{and} \quad P[\bar{\alpha}_0^{L_1} = 1] = 1-q; \;\; P[\bar{\alpha}_0^{L_2} = \frac{1}{2}] = 1. \tag{11}$$

Since $P^0_{L_i}$ and P have the same null-sets on $\mathcal{I}_{L_i}$, we can conclude that Φ is not pseudo-L_1-ergodic or pseudo-L_3-ergodic, but it is pseudo-L_2-ergodic. Since P and $Q^0_{L_i}$ agree on $\mathcal{I}_{L_i}$, we may equivalently replace P in (3.11) by $Q^0_{L_1}$ in the first two expressions, and by $Q^0_{L_2}$ in the last. By the first part of Theorem 2.2 it follows that

$$P^0_{L_1}[\bar{\alpha}_0^{L_1} = \frac{1}{3}] = \frac{3q}{\lambda(L_1)} \quad \text{and} \quad P^0_{L_1}[\bar{\alpha}_0^{L_1} = 1] = \frac{1-q}{\lambda(L_1)},$$

where $\lambda(L_1) = q\lambda_1(L_1) + (1-q)\lambda_2(L_1) = 1 + 2q$ is the intensity of the L_1-claims.

More generally, it can be proved that

$$\begin{aligned} P^0_{L_1}(A) &= \tfrac{3q}{1+2q} P^0_{1,L_1}(A) + \tfrac{1-q}{1+2q} P^0_{2,L_1}(A), \\ Q^0_{L_1}(A) &= qQ^0_{1,L_1}(A) + (1-q)Q^0_{2,L_1}(A), \end{aligned}$$

$A \in \mathcal{M}^\infty_{L_1}$. (Here P^0_{j,L_1} and Q^0_{j,L_1} denote the ordinary and the modified Palm distribution of P_j with respect to L_1.) This is another reason why we believe that modified Palm distributions are more natural than the ordinary Palm distributions. By Lemma

3.4 it follows immediately that the long-run average number of L_3-claims between two successive L_1-claims is equal to 1/3 or 3 with probabilities q and $1-q$, respectively, under both P and $Q^0_{L_1}$. Under $P^0_{L_1}$ the corresponding probabilities are $3q/(1+2q)$ and $(1-q)/(1+2q)$.

4 APPROXIMATION OF P STARTING FROM P^0_L

In Glynn and Sigman [8] convergence is considered for Cesàro means, uniform over functions f with $|f|$ bounded by a fixed function g. In the context of synchronous processes associated with a point process on $[0,\infty)$ sufficient conditions are formulated in Theorem 3.1 of this reference. In the ergodic case the theorem is applied to continuous time Harris recurrent Markov processes and, more specifically, to open Jackson queueing networks. In the present section we derive necessary and sufficient conditions for similar results within the framework of marked point processes on $\mathbb{R}$, using techniques which follow from Theorem 2.1. The Cesàro means $t^{-1}\int_0^t E^0_L(f\circ T_x)dx$ and $t^{-1}\int_0^t P^0_L[T_x\varphi\in B]dx$ will be considered. By the limits (as $t\to\infty$) of the means of the latter type another time-stationary distribution Q_L is defined which in a sense gives a better fit to the intuitive definition (1.1) than P itself. The relationship between Q_L, P, Q^0_L and P^0_L is investigated.

By time-stationarity of Φ, the ergodic theorem yields that for suitable functions f the possibly random limit of $t^{-1}\int_0^t f\circ T_x dx$ exists P-a.s. This limit can be characterized as a conditional expectation on the invariant σ-field $\mathcal{I}_L$. Since the restrictions of P and P^0_L to $\mathcal{I}_L$ have the same null-sets, we have the following so-called cross ergodic result:

$$\frac{1}{t}\int_0^t f\circ T_x dx \to E(f|\mathcal{I}_L) \quad P\text{ -and } P^0_L\text{-a.s.} \tag{1}$$

for all functions $f: M^\infty_L \to \mathbb{R}$ with $E|f|<\infty$. See also Theorem 3 in Nieuwenhuis [18]. The limit $E(f|\mathcal{I}_L)$ equals Ef if Φ is ergodic. If $(t^{-1}\int_0^t f\circ T_x dx)_{t\geq 1}$ is u.i. under P^0_L, then

$$\frac{1}{t}\int_0^t E^0_L(f\circ T_x)dx \to E^0_L(E(f|\mathcal{I}_L)). \tag{2}$$

In this case we obtain for the choice $f(\varphi)=\varphi((0,1]\times L)$:

$$\frac{1}{t}\int_0^t E^0_L(N_L(x+1)-N_L(x))dx \to E^0_L(E(N_L(1)|\mathcal{I}_L)). \tag{3}$$

(Recall the definition of $N_L(1)$ preceding Lemma 3.3 and note that $N_L(x+1) - N_L(x) = N_L(1) \circ T_x$, the number of L-points in the interval $(x, x+1]$.) By the intuitive definition (1.1) of P it might be expected that the limit in (4.3) is equal to $EN_L(1) = \lambda(L)$. However, by (2.4), conditioning on $\mathcal{I}_L$, and Lemma 3.3 we obtain:

$$\begin{aligned} E_L^0(E(N_L(1)|\mathcal{I}_L)) &= \frac{1}{\lambda(L)} E\left(\frac{1}{\alpha_0^L} E(N_L(1)|\mathcal{I}_L)\right) = \frac{1}{\lambda(L)} E[(E(N_L(1)|\mathcal{I}_L))^2] \\ &\geq \frac{1}{\lambda(L)} (EN_L(1))^2 = \lambda(L). \end{aligned}$$

Equality holds iff Φ is pseudo-L-ergodic. We conclude that for a formalization of (1.1) without any ergodicity restraint, we have to be careful because $E_L^0(E(f|\mathcal{I}_L))$ is not necessarily equal to Ef. It is, however, possible to write $E_L^0(E(f|\mathcal{I}_L))$ as an expectation of f. Let the probability measure Q_L on $(M_L^\infty, \mathcal{M}_L^\infty)$ be defined by

$$Q_L(B) := E_L^0[E(1_B|\mathcal{I}_L)], \quad B \in \mathcal{M}_L^\infty.$$

By (4.1) it follows that

$$\frac{1}{t}\int_0^t P_L^0[T_x\varphi \in B]dx \to Q_L(B)$$

for all $B \in \mathcal{M}_L^\infty$. We can interpret $Q_L(B)$ as the expected value (under P_L^0) of the long-run proportion of time points x where "B is seen". Q_L arises from P_L^0 by shifting the origin to a time point chosen at random in $(0, t]$ and letting t tend to infinity. This means that Q_L fits in nicely with (1.1), not P. Replacing B in the above convergence by $[T_a\varphi \in B]$ yields that $Q_L(B) = Q_L[T_a\varphi \in B]$ for all $a \in \mathbb{R}$. Hence, Q_L is also time-stationary. We call it a *modified* time-stationary distribution. Note also that $Q_L = P_L^0$ on $\mathcal{I}_L$. By Theorem 2.1(ii) and conditioning on $\mathcal{I}_L$ we obtain

$$Q_L(B) = \frac{1}{\lambda(L)} E\left[\frac{1}{\alpha_0^L} E(1_B|\mathcal{I}_L)\right] = \frac{1}{\lambda(L)} E\left[1_B E\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right)\right].$$

Since $E(1/\alpha_0^L|\mathcal{I}_L) > 0$ P-a.s.,

$$Q_L \sim P \qquad \text{and} \qquad \frac{dQ_L}{dP} = \frac{1}{\lambda(L)} E\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right) \qquad P\text{-a.s.} \tag{4}$$

Consequently, $E_{Q_L} f = E(fE(1/\alpha_0^L|\mathcal{I}_L))/\lambda(L) = E_L^0(Ef|\mathcal{I}_L))$. So, the limit in (4.2) is equal to $E_{Q_L} f$.

Uniform integrability will be the main condition to obtain limit results as in (4.2). For nonnegative functions f we can transform uniform P_L^0-integrability of the family $(t^{-1}\int_0^t f\circ T_x dx)_{t\geq 1}$ into uniform P-integrability for a similar family of r.v.'s.

lemma 4.1 *Let $g : M_L^\infty \to [0,\infty)$ be P-integrable. Then:*

$$\left(\frac{1}{t}\int_0^t g\circ T_x dx\right)_{t\geq 1} \textit{u.i. under} \quad P_L^0 \iff \left(\frac{1}{\alpha_0^L}\frac{1}{t}\int_0^t g\circ T_x dx\right)_{t\geq 1} \textit{u.i. under} \quad P$$
$$\iff \left(g\frac{1}{t}\int_0^t \frac{1}{\alpha_0^L\circ T_{-x}} dx\right)_{t\geq 1} \textit{u.i. under} \quad P.$$

Proof. It is an easy exercise to prove that under P_L^0 uniform integrability of the family $\left(t^{-1}\int_0^t g\circ T_x dx\right)_{t\geq 1}$ is equivalent to uniform integrability of the sequence $(n^{-1}\int_0^n g\circ T_x dx)_{n\in\mathbb{N}}$. By Lemma 2.1, (4.1) with f replaced by g, and (2.4) we obtain:

$$\left(\tfrac{1}{t}\textstyle\int_0^t g\circ T_x dx\right)_{t\geq 1} \text{ u.i. under } P_L^0$$

$$\iff \begin{cases} E_L^0(E(g|\mathcal{I}_L)) < \infty, \;\; E\left(\frac{1}{n\alpha_0^L}\int_{X_0^L}^{X_0^L+n} g\circ T_x dx\right) < \infty \text{ for all } n\in\mathbb{N}, \\ \frac{1}{n\lambda(L)} E\left(\frac{1}{\alpha_0^L}\int_{X_0^L}^{X_0^L+n} g\circ T_x dx\right) \to E_L^0(E(g|\mathcal{I}_L)). \end{cases}$$

Note that, by (2.3) and time-stationarity,

$$\begin{aligned}
&\frac{1}{n\lambda(L)} E\left|\frac{1}{\alpha_0^L}\int_{X_0^L}^{X_0^L+n} g\circ T_x dx - \frac{1}{\alpha_0^L}\int_0^n g\circ T_x dx\right| \leq \\
&\leq \frac{1}{n\lambda(L)}\Bigg\{E\left[\frac{1}{\alpha_0^L}\int_{X_0^L}^0 g\circ T_x dx\mathbf{1}[X_0^L+n<0]\right] + E\left[\frac{1}{\alpha_0^L}\int_{X_0^L+n}^n g\circ T_x dx\mathbf{1}[X_0^L+n<0]\right] \\
&\qquad + E\left[\frac{1}{\alpha_0^L}\int_{X_0^L}^0 g\circ T_x dx\mathbf{1}[X_0^L+n\geq 0]\right] + E\left[\frac{1}{\alpha_0^L}\int_{X_0^L+n}^n g\circ T_x dx\mathbf{1}[X_0^L+n\geq 0]\right]\Bigg\} \\
&\leq \frac{1}{n\lambda(L)}\left\{E\left[\frac{1}{\alpha_0^L}\int_{X_0^L}^{X_1^L} g\circ T_x dx\right] + E\left[\frac{1}{\alpha_0^L}\int_{X_0^L+n}^{X_1^L+n} g\circ T_x dx\right]\right\} \\
&= \frac{1}{n}\left\{E_L^0\int_0^{\alpha_0^L} g\circ T_x dx + E_L^0\int_0^{\alpha_0^L} g\circ T_n\circ T_x dx\right\} = \frac{1}{n\lambda(L)}\{Eg + Eg\circ T_n\} \\
&= \frac{2}{n\lambda(L)} Eg.
\end{aligned}$$

Since $Eg < \infty$ it follows that the right-hand part of the above equivalence is in turn equivalent to

$$\begin{cases} E_L^0(E(g|\mathcal{I}_L)) < \infty, \quad E\left(\frac{1}{n\alpha_0^L}\int_0^n g \circ T_x dx\right) < \infty \text{ for all } n \in \mathbb{N}, \\ \frac{1}{n\lambda(L)} E\left(\frac{1}{\alpha_0^L}\int_0^n g \circ T_x dx\right) \to E_L^0(E(g|\mathcal{I}_L)). \end{cases}$$

By Lemma 2.1 the first equivalence of the present lemma follows immediately. Since

$$\frac{1}{t}\int_0^t \frac{g \circ T_x}{\alpha_0^L} dx \to \frac{E(g|\mathcal{I}_L)}{\alpha_0^L} \text{ and } \frac{1}{t}\int_0^t \frac{g}{\alpha_0^L \circ T_{-x}} dx \to gE\left(\frac{1}{\alpha_0^L}|\mathcal{I}_L\right) \quad P\text{-a.s.},$$

the second equivalence is also a consequence of Lemma 2.1 (use Fubini's theorem, time-stationarity of P, and conditioning on $\mathcal{I}_L$). □

In the following theorem $\sup_{|f|\leq g}$ means the supremum over all measurable functions $f : M_L^\infty \to \mathbb{R}$ with $|f| \leq g$. Recall the definition of pseudo-L-ergodicity in (2.8).

theorem 4.1 *Let $g : M_L^\infty \to [0,\infty)$ be P-integrable. Then $(t^{-1}\int_0^t g \circ T_x dx)_{t\geq 1}$ is uniformly P_L^0-integrable iff $E_L^0(E(g|\mathcal{I}_L)) < \infty$, $E_L^0(g \circ T_x) < \infty$ for almost every $x > 0$, and*

$$\sup_{|f|\leq g}\left|\frac{1}{t}\int_0^t E_L^0(f \circ T_x)dx - E_{Q_L}f\right| \to 0 \text{ as } t \to \infty. \tag{5}$$

If Φ is pseudo-L-ergodic, then the limits $E_{Q_L}f$ are equal to Ef.

Proof. First the only if-part of the iff statement. The finiteness of the expectations follows from Lemma 2.1 and Fubini's theorem. By Theorem 2.1 we have

$$\frac{1}{t}\int_0^t E_L^0(f \circ T_x)dx = \frac{1}{\lambda(L)t}\int_0^t E\left(\frac{1}{\alpha_0^L} f \circ T_x \circ \theta_{0,L}\right)dx.$$

So, to prove (4.5) it is sufficient to prove that (4.6) and (4.7) below are satisfied:

$$\sup_{|f|\le g} \frac{1}{\lambda(L)t}\left|\int_0^t E(\frac{1}{\alpha_0^L} f\circ T_x \circ \theta_{0,L})dx - \int_0^t E(\frac{1}{\alpha_0^L} f\circ T_x)dx\right| \to 0, \tag{6}$$

$$\sup_{|f|\le g} \left|\frac{1}{\lambda(L)t}\int_0^t E(\frac{1}{\alpha_0^L} f\circ T_x)dx - E_{Q_L} f\right| \to 0, \tag{7}$$

as $t \to \infty$. By considering the expression below successively on $[X_0^L + n < 0]$ and $[X_0^L + n \ge 0]$ as in the proof of Lemma 4.1, we obtain:

$$\frac{1}{\lambda(L)t\alpha_0^L}\left|\int_0^t f\circ T_x \circ \theta_{0,L}dx - \int_0^t f\circ T_x dx\right| \le \frac{1}{\lambda(L)t\alpha_0^L}\left\{\int_{X_0^L}^{X_1^L} g\circ T_x dx + \int_{X_0^L+t}^{X_1^L+t} g\circ T_x dx\right\}$$

for all functions $f : M_L^\infty \to \mathbb{R}$ with $|f| \le g$. This upper bound does not depend on f. So, the supremum in (4.6) is bounded from above by

$$\frac{1}{\lambda(L)t}E(\frac{1}{\alpha_0^L}\int_0^{\alpha_0^L} g\circ T_x\circ\theta_{0,L}dx) + \frac{1}{\lambda(L)t}E(\frac{1}{\alpha_0^L}\int_t^{\alpha_0^L+t} g\circ T_x\circ\theta_{0,L}dx) = \frac{2}{\lambda(L)t}Eg.$$

Again arguments as in the proof of Lemma 4.1 are used here. Relation (4.6) follows immediately. Next (4.7). By Theorem 2.1 and time-stationarity of P we have

$$\left|\frac{1}{\lambda(L)t}\int_0^t E(\frac{1}{\alpha_0^L} f\circ T_x)dx - E_{Q_L}f\right| = \frac{1}{\lambda(L)}\left|\frac{1}{t}\int_0^t E(f\frac{1}{\alpha_0^L\circ T_{-x}})dx - E\left(fE(\frac{1}{\alpha_0^L}\,|\mathcal{I}_L)\right)\right|$$
$$\le \frac{1}{\lambda(L)}E\left[g\left|\frac{1}{t}\int_0^t \frac{1}{\alpha_0^L\circ T_{-x}}dx - E(\frac{1}{\alpha_0^L}|\mathcal{I}_L)\right|\right].$$

This upper bound tends to zero because of the second equivalence in Lemma 4.1. Relation (4.7) follows.

The if-part of the iff statement follows immediately from (4.1) (with f replaced by g) and Lemma 2.1. The last part of the theorem is a consequence of (4.4). □

Let $g : M_L^\infty \to [0,\infty)$ be P-integrable. By the well-known (ε,δ)-characterization of uniform integrability (see, e.g., T25 on p. 286 in Brémaud [5]) and Lemma 4.1 the following implications are obvious:

$$\begin{array}{llll} (g\circ T_x)_{x>0} & \text{u.i. under } P_L^0 & \Longrightarrow \left(t^{-1}\int_0^t g\circ T_x dx\right)_{t\ge 1} & \text{u.i. under } P_L^0, \\ \left(\frac{1}{\alpha_0^L}g\circ T_x\right)_{x>0} & \text{u.i. under } P & \Longrightarrow \left(t^{-1}\int_0^t g\circ T_x dx\right)_{t\ge 1} & \text{u.i. under } P_L^0. \end{array} \tag{8}$$

Note also that

$$
\begin{aligned}
\sup_{x>0} E\left(\frac{1}{\alpha_0^L} g\circ T_x \mathbf{1}[\frac{1}{\alpha_0^L} g\circ T_x > b]\right) &\leq \sup_{x>0} E\left(\frac{1}{\alpha_0^L} g\circ T_x \mathbf{1}[\frac{1}{\alpha_0^L} > \sqrt{b}]\right) + \\
&+ \sup_{x>0} E\left(\frac{1}{\alpha_0^L} g\circ T_x \mathbf{1}[g\circ T_x > \sqrt{b}]\right) \\
\leq \sqrt{E\left[\left(\frac{1}{\alpha_0^L}\right)^2 \mathbf{1}[\left(\frac{1}{\alpha_0^L}\right)^2 > b]\right]}\sqrt{Eg^2} &+ \sqrt{E\left(\frac{1}{\alpha_0^L}\right)^2}\sqrt{E(g^2\mathbf{1}[g^2 > b])}.
\end{aligned}
$$

Consequently,

Corollary 4.1 *Suppose that* $E[(1/\alpha_0^L)^2] < \infty$. *Let* $g : M_L^\infty \to [0,\infty)$ *be such that* $Eg^2 < \infty$. *Then*

$$\sup_{|f|\leq g}\left|\frac{1}{t}\int_0^t E_L^0(f\circ T_x)dx - E_{Q_L}f\right| \to 0 \ \text{as}\ t\to\infty.$$

When starting from P_L^0, we can consider Q_L as the uniform limit (as $t \to \infty$) of the distribution of the MPP seen from a position chosen at random in the interval $(0, t]$. The limit Q_L is equal to P if $n^{-1}\sum_{i=1}^n \alpha_i^L \to 1/\lambda(L) \quad P_L^0$-a.s. These assertions are expressed in the following corollary. It is an immediate consequence of Theorem 4.1.

Corollary 4.2 *The convergence*

$$\frac{1}{t}\int_0^t P_L^0[T_x\varphi \in B]dx \to Q_L(B) \tag{9}$$

holds uniformly over $B \in \mathcal{M}_L^\infty$. $Q_L = P$ *iff* Φ *is pseudo-L-ergodic.*

The existence of the limit in (4.9) was already proved in Satz 2.1 in Nawrotzki [14].

For an MPP Φ and a set L of marks, we introduced (under the assumptions of Section 2) the time-stationary distributions P and Q_L, and the L-event-stationary distributions P_L^0 and Q_L^0. All these distributions have the same null-sets on $\mathcal{I}_L$. By stationarity considerations,

$$\frac{1}{n}\sum_{i=1}^{n} 1_B \circ \theta_{i,L} \to P_L^0(B|\mathcal{I}_L) \text{ a.s. and } \quad \frac{1}{t}\int_0^t 1_B \circ T_x dx \to P(B|\mathcal{I}_L) \text{ a.s.,}$$

$B \in \mathcal{M}_L^\infty$. Taking expectations of the left-hand side (LHS) under P and Q_L, respectively, and noting that $P_L^0(B|\mathcal{I}_L) = Q_L^0(B|\mathcal{I}_L)$ a.s. (see (2.12)), that $P = Q_L^0$ on $\mathcal{I}_L$, and that $Q_L = P_L^0$ on $\mathcal{I}_L$, yields

$$\frac{1}{n}\sum_{i=1}^{n} P[\theta_{i,L}\varphi \in B] \to Q_L^0(B) \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n} Q_L[\theta_{i,L}\varphi \in B] \to P_L^0(B) \tag{10}$$

for all $B \in \mathcal{M}_L^\infty$. Similarly, taking expectations of the right-hand side (RHS) under P_L^0 and Q_L^0, and noting that $P(B|\mathcal{I}_L) = Q_L(B|\mathcal{I}_L)$ a.s., that $P_L^0 = Q_L$ on $\mathcal{I}_L$, and that $Q_L^0 = P$ on $\mathcal{I}_L$, yields

$$\frac{1}{t}\int_0^t P_L^0[T_x\varphi \in B]dx \to Q_L(B) \quad \text{and} \quad \frac{1}{t}\int_0^t Q_L^0[T_x\varphi \in B]dx \to P(B) \tag{11}$$

for all $B \in \mathcal{M}_L^\infty$. (Only the (pointwise) convergences on the right in (4.10) and (4.11) are new (cf. Theorem 2.2 and Corollary 4.2); see Section 7 for uniform versions.) In the spirit of (1.2) and (1.1) one might (in the general, not necessarily pseudo-L-ergodic, case) say that

> Q_L^0 and P_L^0 are the distributions seen from an L -point chosen at random when starting from P and Q_L, respectively;
>
> Q_L and P are the distributions seen from a time-point chosen at random when starting from P_L^0 and Q_L^0, respectively.

Apart from the LHS of (4.10), also the RHS of (4.11) illustrates that in a sense the **modified** Palm distribution is more closely related to the time-stationary distribution than the ordinary one. Note also that, by the RHS of (4.10), P_L^0 is the modified Palm distribution of Q_L if we assume additionally that $E[(E(N_L(1)|\mathcal{I}_L))^2] < \infty$ (and hence the intensity $E_{Q_L}(N_L(1))$ is finite; see (4.4)).

5 APPROXIMATION OF P_L^0 STARTING FROM P

When starting from P, the distribution of Φ seen from an L-point chosen at random from the first n L-points tends uniformly to Q_L^0 as $n \to \infty$; see Theorem 2.2. In the present section we generalize this result to a uniform limit theorem for Cesàro averaged functionals $(n^{-1}\sum_{i=1}^n Ef\circ\theta_{i,L})_{n\in\mathbb{N}}$.

Consider some function $f : M_L^\infty \to \mathbb{R}$ with $E_L^0|f| < \infty$. Since P_L^0 is L-event-stationary, the (possibly random) limit of $n^{-1}\sum_{i=1}^n f\circ\theta_{i,L}$ exists P_L^0-a.s. and can be characterized as a conditional expectation. The following cross ergodic result is a direct consequence of the fact that P and P_L^0 have the same null-sets on $\mathcal{I}_L$:

$$\frac{1}{n}\sum_{i=1}^n f\circ\theta_{i,L} \to E_L^0(f|\mathcal{I}_L) \quad P_L^0\text{ - and } P\text{- a.s.} \tag{1}$$

Note that the limit is equal to $E_L^0 f$ if Φ is ergodic. If $(n^{-1}\sum_{i=1}^n f\circ\theta_{i,L})_{n\geq 1}$ is u.i. under P, then

$$\frac{1}{n}\sum_{i=1}^n Ef\circ\theta_{i,L} \to E(E_L^0(f|\mathcal{I}_L)). \tag{2}$$

Because of (2.5) and (2.10) it is an easy exercise to prove that the limit in (5.2) is equal to $E_{Q_L^0} f$.

The main condition in Theorem 5.1 below is about uniform P-integrability of $(n^{-1}\sum_{i=1}^n g\circ\theta_{i,L})_{n\geq 1}$. In the following lemma this is characterized. It will be applied in the proof of the theorem.

lemma 5.1 *Let $g : M_L^\infty \to [0,\infty)$ be P_L^0-integrable. Then*

$$\begin{aligned}\left(\frac{1}{n}\sum_{i=1}^n g\circ\theta_{i,L}\right)_{n\geq 1} \textit{ u.i. under } P &\iff \left(\alpha_0^L\frac{1}{n}\sum_{i=1}^n g\circ\theta_{i,L}\right)_{n\geq 1} \textit{ u.i. under } P_L^0\\ &\iff \left(g\frac{1}{n}\sum_{i=1}^n \alpha_{-i}^L\right)_{n\geq 1} \textit{ u.i. under } P_L^0.\end{aligned}$$

Proof. By (5.1) and Lemma 2.1, (2.5), and (2.2) we obtain:

$$\left(\frac{1}{n}\sum_{i=1}^n g\circ\theta_{i,L}\right)_{n\geq 1} \quad \text{u.i. under } P$$

$$\iff \begin{cases} E(E_L^0(g|\mathcal{I}_L)) < \infty, \ Eg\circ\theta_{i,L} < \infty \text{ for all } i \in \mathbb{N}, \text{ and} \\ \frac{1}{n}\sum_{i=1}^n Eg\circ\theta_{i,L} \to E(E_L^0(g|\mathcal{I}_L)) \end{cases}$$

$$\iff \begin{cases} E_L^0\left(\alpha_0^L E_L^0(g|\mathcal{I}_L)\right) < \infty, \ E_L^0\left(\alpha_0^L g\circ\theta_{i,L}\right) < \infty \text{ for all } i \in \mathbb{N}, \text{ and} \\ \frac{1}{n}\sum_{i=1}^n E_L^0\left(\alpha_0^L g\circ\theta_{i,L}\right) \to E_L^0\left(\alpha_0^L E_L^0(g|\mathcal{I}_L)\right) \end{cases}$$

$$\iff \begin{cases} E_L^0(g\bar{\alpha}_0^L) < \infty, \ E_L^0(g\alpha_{-i}^L) < \infty \text{ for all } i \in \mathbb{N}, \text{ and} \\ \frac{1}{n}\sum_{i=1}^n E_L^0(g\alpha_{-i}^L) \to E_L^0(g\bar{\alpha}_0^L). \end{cases}$$

Note that

$$\alpha_0^L \frac{1}{n}\sum_{i=1}^n g\circ\theta_{i,L} \to \alpha_0^L E_L^0(g|\mathcal{I}_L) \ \text{ and } \ g\frac{1}{n}\sum_{i=1}^n \alpha_{-i}^L \to g\bar{\alpha}_0^L \quad P_L^0\text{-a.s.} \tag{3}$$

So, by Lemma 2.1 the right-hand parts of the second and third equivalences above are in turn equivalent to uniform P_L^0-integrability of $\left(\alpha_0^L n^{-1}\sum_{i=1}^n g\circ\theta_{i,L}\right)_{n\geq 1}$ and $\left(gn^{-1}\sum_{i=1}^n \alpha_{-i}^L\right)_{n\geq 1}$, respectively. □

The following theorem is a generalization of a part of Theorem 2.2. Here $\sup_{|f|\leq g}$ means the supremum over all measurable functions $f : M_L^\infty \to \mathbb{R}$ with $|f| \leq g$, i.e. with $|f(\varphi)| \leq g(\varphi)$ for all $\varphi \in M_L^\infty$.

theorem 5.1 *Let $g : M_L^\infty \to [0,\infty)$ be P_L^0-integrable. Then $(n^{-1}\sum_{i=1}^n g\circ\theta_{i,L})_{n\geq 1}$ is uniformly P-integrable iff $E(E_L^0(g|\mathcal{I}_L)) < \infty$, $Eg\circ\theta_{i,L} < \infty$ for all $i \in \mathbb{N}$, and*

$$\sup_{|f|\leq g}\left|\frac{1}{n}\sum_{i=1}^n Ef\circ\theta_{i,L} - E_{Q_L^0} f\right| \to 0. \tag{4}$$

If Φ is pseudo-L-ergodic, then the limits $E_{Q_L^0} f$ are equal to $E_L^0 f$.

Proof. The last part follows immediately, since $E_{Q_L^0} f = \lambda(L) E_L^0\left(\bar{\alpha}_0^L f\right)$. Suppose that $(n^{-1}\sum_{i=1}^n g\circ\theta_{i,L})$ is u.i. under P. By (5.1) and Lemma 2.1 the finiteness of $E(E_L^0(g|\mathcal{I}_L))$ and $Eg\circ\theta_{i,L}$, $i \in \mathbb{N}$, is obvious. By Theorem 2.1 we obtain

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^{n} Ef\circ\theta_{i,L} - E_{Q_L^0} f\right| &= \left|\frac{1}{n}\sum_{i=1}^{n} E_{i,L} f - E_{Q_L^0} f\right| \\ &= \lambda(L)\left|E_L^0(\frac{1}{n}\sum_{i=1}^{n} f\alpha_{-i}^L) - E_L^0(f\bar{\alpha}_0^L)\right| \\ &\leq \lambda(L)E_L^0\left[g|\frac{1}{n}\sum_{i=1}^{n}\alpha_{-i}^L - \bar{\alpha}_0^L|\right]\end{aligned}$$

for all measurable functions $f : M_L^\infty \to \mathbb{R}$ with $|f| \leq g$. This upper bound does not depend on f, and tends to zero because of the last equivalence in Lemma 5.1. Relation (5.4) follows. The reversed implication of the iff statement is an immediate consequence of (5.1) and Lemma 2.1. □

Note that the proof also leads to an upper bound for the supremum in (5.4); see also Theorem 2.2.

Remark. In view of Section 7 slight generalizations of Lemma 5.1 and Theorem 5.1 are of interest. Apart from $g : M_L^\infty \to [0,\infty)$ with $E_L^0 g < \infty$, an arbitrary (but fixed) $\mathcal{I}_L$-measurable function $\beta : M_L^\infty \to [0,\infty)$ is considered. Since $\beta n^{-1}\sum_{i=1}^n g\circ\theta_{i,L} \to \beta E_L^0(g|\mathcal{I}_L)$ P-a.s., it is an easy exercise to prove that the conclusions of Lemma 5.1 and Theorem 5.1 remain valid if g is replaced by βg and f by βf; $\sup_{|f|\leq g}$ remains unchanged. By these replacements (5.4) turns into (cf. (3.1)):

$$\sup_{|f|\leq g}\left|\frac{1}{n}\sum_{i=1}^{n} E\left(\beta f\circ\theta_{i,L}\right) - E_{Q_L^0}(\beta f)\right| \to 0.$$

Note that the P_L^0-integrability of g (and not of βg) remains the only condition for the validity of the equivalence in Theorem 5.1 when generalized as above.

By the (ε,δ)-characterization of uniform integrability (see p. 286 in Brémaud [5]) it is obvious that

$$\left(g\circ\theta_{i,L}\right)_{i\geq 1} \text{ u.i. under } P \Longrightarrow \left(\frac{1}{n}\sum_{i=1}^{n} g\circ\theta_{i,L}\right)_{n\geq 1} \text{ u.i. under } P. \tag{5}$$

Note also that

$$E(g \circ \theta_{i,L} \mathbf{1}[g \circ \theta_{i,L} > b]) = \lambda(L) E_L^0(\alpha_{-i}^L g \mathbf{1}[g > b]) \leq \lambda(L) \sqrt{E_L^0[(\alpha_0^L)^2] E_L^0[g^2 \mathbf{1}[g > b]]},$$

which tends to zero as $b \to \infty$, provided that $E_L^0(\alpha_0^L)^2$ and $E_L^0 g^2$ (or, equivalently, $E\alpha_0^L$ and $E(g^2 \circ \theta_{0,L}/\alpha_0^L)$) are finite. We conclude:

Corollary 5.1 *Suppose that $E_L^0(\alpha_0^L)^2 < \infty$. Let $g : M_L^\infty \to [0, \infty)$ be such that $E_L^0 g^2 < \infty$. Then*

$$\sup_{|f| \leq g} \left| \frac{1}{n} \sum_{i=1}^{n} E f \circ \theta_{i,L} - E_{Q_L^0} f \right| \to 0 \quad \text{as } n \to \infty.$$

6 APPROXIMATION OF P_L^0 STARTING FROM $P_{L'}^0$

In this section two nonempty sets of marks, L and L', are considered. We define a probability measure which intuitively arises from $P_{L'}^0$ by shifting the origin to an L-point chosen at random. For the case that P is replaced by $P_{L'}^0$ results similar to the results of Section 5 are derived.

Let $L, L' \in \operatorname{Bor} K$ be nonempty and $P(M_{L,L'}^\infty) = 1$. When two sets of marks are involved, we will always restrict $\theta_{i,L}, \theta_{i,L'}$, and T_t to $M_{L,L'}^\infty$. By Lemma 3.2 the invariant σ-fields for these families of shifts coincide, so we write $\mathcal{I}$ instead of $\mathcal{I}_{L,L'}, \mathcal{I}_{L',L}$ and $\mathcal{I}'_{L,L'}$. We will prove a theorem similar to Theorem 5.1 in the case that P is replaced by $P_{L'}^0$. Some preliminaries are needed first. Recall the definition of the random variables $\xi_i, i \in \mathbb{Z}$, in (3.5), and let similar random variables η_i be defined by

$$\eta_i(\varphi) := \varphi([X_i^L(\varphi), X_{i+1}^L(\varphi)) \times L'), \quad \varphi \in M_{L,L'}^\infty, \tag{1}$$

the number of L'-points in the interval $[X_i^L(\varphi), X_{i+1}^L(\varphi))$. Note that

$$\xi_i \circ \theta_{j,L} = \xi_{i+j} \quad \text{and} \quad \eta_i \circ \theta_{j,L} = \eta_{i+j} \tag{2}$$

for all $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$, and that ξ_i can be different from η_i if L and L' are not disjoint. The following theorem is the analogue of Theorem 2.1 for the case that P is replaced by $P^0_{L'}$. It enables us to write $P^0_{L'}$-expectations of a special class of random variables as simple P^0_L-expectations.

theorem 6.1 *Let $n \in \mathbb{Z}$. Then*

(i) $P^0_{L'}\theta^{-1}_{n,L} << P^0_L$,

(ii) $\dfrac{d(P^0_{L'}\theta^{-1}_{n,L})}{dP^0_L} = \dfrac{\lambda(L)}{\lambda(L')}\eta_{-n}$ P^0_L-a.s.

Proof. Set $\zeta_0(\varphi) := \varphi((X^L_0(\varphi), X^L_1(\varphi)) \times L')$ and note that the sequence $(k^L_i)_{i\in Z}$ of marks satisfies $k^L_i \circ \theta_{j,L} = k^L_{i+j}$ for all $i, j \in \mathbb{Z}$. By (3.10) we obtain

$$\begin{aligned} P^0_{L'}[\theta_{0,L}\varphi \in A] &= \frac{\lambda(L)}{\lambda(L')}E^0_L\left[\textstyle\sum_{i=1}^{\xi_0} 1_A \circ \theta_{0,L} \circ \theta_{i,L'}\right] \\ &= \frac{\lambda(L)}{\lambda(L')}E^0_L[\zeta_0 1_A \circ \theta_{0,L} + \mathbf{1}[k^L_1 \in L']1_A \circ \theta_{1,L}] \\ &= \frac{\lambda(L)}{\lambda(L')}[E^0_L(\zeta_0 1_A) + E^0_L(\mathbf{1}[k^L_0 \in L']1_A)] \\ &= \frac{\lambda(L)}{\lambda(L')}E^0_L[(\zeta_0 + \mathbf{1}[k^L_0 \in L'])1_A] = \frac{\lambda(L)}{\lambda(L')}E^0_L(\eta_0 1_A). \end{aligned}$$

This completes the proof for $n = 0$. For general $n \in \mathbb{Z}$, replace A in the above by $[\theta_{n,L}(\varphi) \in A]$ and apply the right-hand part of (6.2). $\square$

Note that $P^0_{L'}\theta^{-1}_{n,L}[\eta_{-n} > 0] = P^0_{L'}[\eta_0 > 0] = 1$ and that

$$P^0_L[\eta_{-n} > 0] = P^0_L[\eta_0 > 0] = \frac{\lambda(L')}{\lambda(L)}E^0_{L'}\left(\frac{1}{\eta_0}\right) \leq \frac{\lambda(L')}{\lambda(L)}.$$

So, P^0_L and $P^0_{L'}\theta^{-1}_{n,L}$ are not necessarily equivalent. As an immediate consequence of Theorem 6.1 we obtain (take $A = M^\infty_{L,L'}$ in the proof and in (3.10), respectively)

$$E^0_L\eta_n = \frac{\lambda(L')}{\lambda(L)} = E^0_L\xi_n, \quad n \in \mathbb{Z}. \tag{3}$$

See also (3.4.2) in Baccelli and Brémaud [2]. Since $E^0_L(\xi_0|\mathcal{I})$ and $E^0_L(\eta_0|\mathcal{I})$ are the a.s. limits (under P^0_L) of $n^{-1}\sum_{i=0}^{n-1}\xi_i$ and $n^{-1}\sum_{i=0}^{n-1}\eta_i$ respectively, it is obvious that

$$E^0_L(\xi_0|\mathcal{I}) = E^0_L(\eta_0|\mathcal{I}) \ \ P^0_L\text{-a.s.}$$

In the following we will write $\bar{\eta}_0$ for $E^0_L(\eta_0|\mathcal{I})$, and $\bar{\alpha}^L_0$ and $\bar{\alpha}^{L'}_0$ for $E^0_L(\alpha^L_0|\mathcal{I})$ and $E^0_{L'}(\alpha^{L'}_0|\mathcal{I})$, respectively. It is an easy exercise to prove that $P^0_{L'}$ and P^0_L are equivalent on $\mathcal{I}$. As a consequence we obtain by Lemma 3.4 the following relation between the above conditional expectations:

$$\bar{\eta}_0 = \frac{\bar{\alpha}^L_0}{\bar{\alpha}^{L'}_0} \quad P^0_L\text{- and } P^0_{L'}\text{-a.s.}$$

Recall (5.1). Since $P^0_{L'}\theta^{-1}_{0,L} << P^0_L$ it is obvious that this convergence holds $P^0_{L'}$-a.s. as well:

$$\frac{1}{n}\sum_{i=1}^{n} f \circ \theta_{i,L} \to E^0_L(f|\mathcal{I}) \quad P^0_{L'}\text{-a.s.} \tag{4}$$

for all P^0_L-integrable functions $f : M^\infty_{L,L'} \to \mathbb{R}$. If $(n^{-1}\sum_{i=1}^n f \circ \theta_{i,L})_{n\geq 1}$ is u.i. under $P^0_{L'}$, then

$$\frac{1}{n}\sum_{i=1}^{n} E^0_{L'} f \circ \theta_{i,L} \to E^0_{L'}(E^0_L(f|\mathcal{I})). \tag{5}$$

The limit in (6.5) can be written as an expectation of f under a special probability measure. Let the probability measure $Q^0_{L,L'}$ be defined by

$$Q^0_{L,L'}(B) := E^0_{L'}(E^0_L(1_B|\mathcal{I})), \quad B \in \mathcal{M}^\infty_{L,L'}. \tag{6}$$

It is an easy exercise to prove that $Q^0_{L,L'}$ is invariant under the set $\{\theta_{n,L} : n \in \mathbb{Z}\}$ of point shifts, i.e.

$$Q^0_{L,L'}\theta^{-1}_{n,L} = Q^0_{L,L'}, \quad n \in \mathbb{Z},$$

but typically not under $\{\theta_{n,L'} : n \in \mathbb{Z}\}$. Let M^0 be the set of φ's in $M^\infty_{L,L'}$ with an L-point in the origin. Then the conditional probability $P^0_L(M^0|\mathcal{I})$ is a.s. equal to 1 under P^0_L, and hence by Theorem 6.1 also under $P^0_{L'}$. Consequently, $Q^0_{L,L'}(M^0) = 1$. Note that

$$
\begin{aligned}
Q^0_{L,L'}(B) &= E^0_{L'}(E^0_L(1_B|\mathcal{I})\circ\theta_{0,L}) = \frac{\lambda(L)}{\lambda(L')}E^0_L(E^0_L(1_B|\mathcal{I})\eta_0) \\
&= \frac{\lambda(L)}{\lambda(L')}E^0_L(1_B\bar{\eta}_0),
\end{aligned}
$$

$B \in \mathcal{M}^\infty_{L,L'}$. Consequently, on $(M^\infty_{L,L'}, \mathcal{M}^\infty_{L,L'})$,

$$
Q^0_{L,L'} \sim P^0_L \quad \text{and} \quad \frac{dQ^0_{L,L'}}{dP^0_L} = \frac{\lambda(L)}{\lambda(L')}\bar{\eta}_0 = \frac{\lambda(L)}{\lambda(L')}\frac{\bar{\alpha}^L_0}{\bar{\alpha}^{L'}_0}. \tag{7}
$$

(Note also that $Q^0_{L,L'} = Q^0_L$ if Φ is pseudo-L'-ergodic; cf. (2.10).) The limit $E^0_{L'}(E^0_L(f|\mathcal{I}))$ in (6.5) is equal to

$$
\frac{\lambda(L)}{\lambda(L')}E^0_L(\eta_0 E^0_L(f|\mathcal{I})) = \frac{\lambda(L)}{\lambda(L')}E^0_L(f\bar{\eta}_0) = E_{Q^0_{L,L'}}f.
$$

Next we state the analogue of Lemma 5.1. Apart from replacing P by $P^0_{L'}$, and α^L_0 by η_0, its proof is similar to the proof of Lemma 5.1. Theorem 6.1 and, again, Lemma 2.1 supply important ingredients.

lemma 6.1 *Let $g : M^\infty_{L,L'} \to [0,\infty)$ be P^0_L-integrable. Then:*

$$
\begin{aligned}
\left(\frac{1}{n}\sum_{i=1}^n g\circ\theta_{i,L}\right)_{n\geq 1} \text{ u.i. under } P^0_{L'} &\iff \left(\eta_0\frac{1}{n}\sum_{i=1}^n g\circ\theta_{i,L}\right)_{n\geq 1} \text{ u.i. under } P^0_L \\
&\iff \left(g\frac{1}{n}\sum_{i=1}^n \eta_{-i}\right)_{n\geq 1} \text{ u.i. under } P^0_L.
\end{aligned}
$$

The following theorem is the analogue of Theorem 5.1; $\sup_{|f|\leq g}$ means the supremum over all measurable functions $f : M^\infty_{L,L'} \to \mathbb{R}$ with $|f| \leq g$.

theorem 6.2 *Let $g : M^\infty_{L,L'} \to [0,\infty)$ be P^0_L-integrable. Then $(n^{-1}\sum_{i=1}^n g\circ\theta_{i,L})$ is uniformly $P^0_{L'}$-integrable iff $E^0_{L'}(E^0_L(g|\mathcal{I})) < \infty$, $E^0_{L'}g\circ\theta_{i,L} < \infty$ for all $i \in \mathbb{N}$, and*

$$
\sup_{|f|\leq g}\left|\frac{1}{n}\sum_{i=1}^n E^0_{L'}f\circ\theta_{i,L} - E_{Q^0_{L,L'}}f\right| \to 0. \tag{8}
$$

If Φ is pseudo-L-ergodic and pseudo-L'-ergodic, then the limits $E_{Q^0_{L,L'}}f$ are equal to $E^0_L f$.

Proof. The last part is a consequence of (6.7). Suppose that $(n^{-1}\sum_{i=1}^n g\circ\theta_{i,L})_{n\geq 1}$ is u.i. under $P^0_{L'}$. For all measurable $f: M^\infty_{L,L'}\to\mathbb{R}$ with $|f|\leq g$ we have (cf. Theorem 6.1 and (6.7)),

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^n E^0_{L'}f\circ\theta_{i,L}-E_{Q^0_{L,L'}}f\right| &= \frac{\lambda(L)}{\lambda(L')}\left|\frac{1}{n}\sum_{i=1}^n E^0_L(f\eta_{-i})-E^0_L(f\bar{\eta}_0)\right|\\ &\leq \frac{\lambda(L)}{\lambda(L')}E^0_L\left[g\left|\frac{1}{n}\sum_{i=1}^n\eta_{-i}-\bar{\eta}_0\right|\right].\end{aligned}$$

This upper bound does not depend on f and tends to zero (as $n\to\infty$) because of Lemma 6.1, which proves (6.8). The reversed implication follows from (6.4) and Lemma 2.1. □

Remarks. Lemma 6.1 and Theorem 6.2 can be generalized slightly by considering, apart from the P^0_L-integrable, nonnegative function g, a fixed $\mathcal{I}$-measurable function $\beta: M^\infty_{L,L'}\to[0,\infty)$. The conclusions of the lemma and the theorem remain valid if g and f are replaced by βg and βf. Relation (6.8) turns into (cf. (3.3))

$$\sup_{|f|\leq g}\left|\frac{1}{n}\sum_{i=1}^n E^0_{L'}(\beta f\circ\theta_{i,L})-E_{Q^0_{L,L'}}(\beta f)\right|\to 0.$$

Again $E^0_L g<\infty$ remains the only assumption. Note also that

$$E^0_{L'}(g\circ\theta_{i,L}\mathbf{1}[g\circ\theta_{i,L}>b])=\frac{\lambda(L)}{\lambda(L')}E^0_L(\eta_{-i}g\mathbf{1}[g>b])\leq\frac{\lambda(L)}{\lambda(L')}\sqrt{E^0_L\eta_0^2E^0_L(g^2\mathbf{1}[g>b])}$$

for all $i\in\mathbb{Z}$. The hypothesis about uniform integrability in Theorem 6.2 is satisfied if $(g\circ\theta_{i,L})_{i\geq 1}$ is u.i. under $P^0_{L'}$, and hence if $E^0_L(\eta_0^2)<\infty$ (or, equivalently, $E^0_{L'}\eta_0<\infty$) and $E^0_L(g^2)<\infty$.

In Konstantopoulos and Walrand [11] weak convergence of the sequence $(P^0_{L'}\theta^{-1}_{n,L})_{n\geq 1}$ of probability measures is considered under some additional mixing condition. See also König and Schmidt [10]. The following corollary of Theorem 6.2 concerns uniform convergence of the sequence $(n^{-1}\sum_{i=1}^n P^0_{L'}\theta^{-1}_{i,L})_{n\geq 1}$ without any additional condition. It expresses that starting with $P^0_{L'}$ we can, as $n\to\infty$, consider $Q^0_{L,L'}$ as the distribution of the MPP seen from an L-point chosen at random among the first n L-points.

Corollary 6.1 *Let $L, L' \in \operatorname{Bor} K$ be nonempty and $P(M^\infty_{L,L'}) = 1$. Then*

$$\sup_{B \in \mathcal{M}^\infty_{L,L'}} \Big|\frac{1}{n}\sum_{i=1}^{n} P^0_{L'}[\theta_{i,L}\varphi \in B] - Q^0_{L,L'}(B)\Big| = \frac{\lambda(L)}{2\lambda(L')} E^0_L \Big|\frac{1}{n}\sum_{i=1}^{n} \eta_{-i} - \bar{\eta}_0\Big|.$$

This supremum tends to 0 as $n \to \infty$.

Proof. By Theorem 6.1 the probability measures $n^{-1}\sum_{i=1}^{n} P^0_{L'}\theta^{-1}_{i,L}$, $n \in \mathbb{Z}$, are all dominated by P^0_L with Radon-Nikodym derivatives $(\lambda(L)/\lambda(L'))n^{-1}\sum_{i=1}^{n}\eta_{-i}$. The equality is an immediate consequence of (2.14) and (6.7). The convergence to 0 follows from Theorem 6.2 with the choice $g \equiv 1$. □

7 APPROXIMATIONS WITHOUT ERGODICITY RESTRAINTS

The intuitive random procedures (1.2) and (1.1) for generating P^0_L and P were formalized in Theorem 2.2 and Corollary 4.2. For a direct approximation of these probability measures a weak ergodicity condition was needed. In this section the results of Sections 4 to 6 will be applied to derive uniform approximations of P^0_L and P without assuming ergodicity properties.

The limits in Theorem 2.2, Corollary 4.2, and Corollary 6.1 are not the distributions P^0_L, P, and P^0_L, but the modified distributions Q^0_L, Q_L, and $Q^0_{L,L'}$, respectively. The pairwise relationships between corresponding probability measures were described by Radon-Nikodym derivatives, which are repeated here:

$$\frac{dQ^0_L}{dP^0_L} = \lambda(L)\bar{\alpha}^L_0, \quad \frac{dQ_L}{dP} = \frac{1}{\lambda(L)\bar{\alpha}^L_0}, \quad \frac{dQ^0_{L,L'}}{dP^0_L} = \frac{\lambda(L)}{\lambda(L')}\bar{\eta}_0. \tag{1}$$

For approximation of P^0_L, starting from P and $P^0_{L'}$ respectively, choices for g and β in the remarks following Theorems 5.1 and 6.2 are suggested by (7.1). Choose, respectively,

$$g \equiv 1 \text{ and } \beta = \frac{1}{\lambda(L)\bar{\alpha}^L_0}, \quad g \equiv 1 \text{ and } \beta = \frac{\lambda(L')}{\lambda(L)}\frac{1}{\bar{\eta}_0}. \tag{2}$$

For g in Theorem 4.1 we take $\lambda(L)\bar{\alpha}_0^L$.

Theorem 7.1

(a) $\sup_{B\in\mathcal{M}_L^\infty}\left|\frac{1}{n}\sum_{i=1}^{n}E\left(\frac{1}{\lambda(L)\bar{\alpha}_0^L}1_B\circ\theta_{i,L}\right)-P_L^0(B)\right|\to 0$ *as* $n\to\infty$.

(b) $\sup_{B\in\mathcal{M}_{L,L'}^\infty}\left|\frac{1}{n}\sum_{i=1}^{n}E_{L'}^0\left(\frac{\lambda(L')}{\lambda(L)}\bar{\eta}_0^{-1}1_B\circ\theta_{i,L}\right)-P_L^0(B)\right|\to 0$ *as* $n\to\infty$.

(c) *If* $E\bar{\alpha}_0^L<\infty$, *then* $\sup_{B\in\mathcal{M}_L^\infty}\left|\frac{1}{t}\int_0^t E_L^0(\lambda(L)\bar{\alpha}_0^L 1_B\circ T_x)dx-P(B)\right|\to 0$ *as* $t\to\infty$.

Proof. For (a) and (b) we choose g and β as suggested in (7.2). By Theorems 2.1 and 6.1 we have:

$$E\left(\frac{1}{\bar{\alpha}_0^L}\right)=\lambda(L)E_L^0\left(\frac{\alpha_0^L}{\bar{\alpha}_0^L}\right)=\lambda(L)\ \text{ and }\ E_{L'}^0\left(\bar{\eta}_0^{-1}\right)=\frac{\lambda(L)}{\lambda(L')}E_L^0\left(\eta_0\bar{\eta}_0^{-1}\right)=\frac{\lambda(L)}{\lambda(L')}.$$

So, the hypotheses about uniform integrability are satisfied since the corresponding sequences contain only one integrable element. By reducing the sets of functions f to the functions 1_B with $B\in\mathcal{M}_L^\infty$ and $B\in\mathcal{M}_{L,L'}^\infty$, respectively, the parts (a) and (b) are immediate consequences of the remarks following Theorems 5.1 and 6.2.

For (c) we apply Theorem 4.1 with $g=\lambda(L)\bar{\alpha}_0^L$. The condition that Eg is finite causes the hypothesis in (c). □

Remarks. By (7.1) the summed expectations in (a) and the integrands in (c) are equal to $Q_L[\theta_{i,L}\varphi\in B]$ and $Q_L^0[T_x\varphi\in B]$, respectively. So, parts (a) and (c) are just uniform versions of the right-hand sides of (4.10) and (4.11). Let η_0' be defined as η_0 in (6.1) with L and L' interchanged. By the equality preceding (6.4) it is obvious that $E_{L'}^0(\eta_0'|\mathcal{I})$ and $E_L^0(\eta_0|\mathcal{I})$ are related as follows:

$$E_{L'}^0\left(\eta_0'|\mathcal{I}\right)=\frac{1}{E_L^0(\eta_0|\mathcal{I})}\quad P_{L'}^0\text{-a.s.}\tag{3}$$

By interchanging L and L' in the right-hand relation in (7.1), it follows that the summed expectations in (b) are equal to $Q_{L',L}^0[\theta_{i,L}\varphi\in B]$.

The finiteness of $E\bar{\alpha}_0^L$ is equivalent to the finiteness of $E_L^0(\bar{\alpha}_0^L)^2$. By Jensen's inequality we have:

$$\left(\bar{\alpha}_0^L\right)^2 \leq E_L^0((\alpha_0^L)^2\,|\mathcal{I}_L) \quad P_L^0\text{-a.s. and } E_L^0\left(\bar{\alpha}_0^L\right)^2 \leq E_L^0\left(\alpha_0^L\right)^2.$$

So, the hypothesis in (c) is satisfied if $E_L^0(\alpha_0^L)^2 < \infty$. All parts of Theorem 7.1 can be generalized to uniform limit results for functions f with $|f| \leq g$, similar to Theorems 5.1, 6.2, and 4.1.

At the end of this section we give interpretations of the results in Theorem 7.1. Note that by Jensen's inequality,

$$E\left(\lambda(L)\bar{\alpha}_0^L\right) = (\lambda(L))^2\,E_L^0\left(\bar{\alpha}_0^L\right)^2 \geq 1 = E_L^0\left(\lambda(L)\bar{\alpha}_0^L\right)$$

(a strict inequality holds in the non-pseudo-L-ergodic case). So, in a transition from P to P_L^0 the importance of realizations φ for which $\lambda(L)\bar{\alpha}_0^L(\varphi)$ is relatively large, should be reconsidered. We conclude that (a) and (c) in Theorem 7.1 can be interpreted as follows:

> P_L^0 arises from P by first changing the weights of the realizations by way of the weight function $1/(\lambda(L)\bar{\alpha}_0^L)$, followed by shifting the origin to an L-point chosen at random from the first n L-points and letting n tend to infinity.

> P arises from P_L^0 by first changing the weights of the realizations by way of the weight function $\lambda(L)\bar{\alpha}_0^L$, followed by shifting the origin to a time point chosen at random in $(0,t)$ and letting t tend to infinity.

By Theorem 6.1, (6.3) and Jensen's inequality, we have:

$$E_{L'}^0\left(\frac{\lambda(L)}{\lambda(L')}\bar{\eta}_0\right) = \left(\frac{\lambda(L)}{\lambda(L')}\right)^2 E_L^0(\bar{\eta}_0^2) \geq 1 = E_L^0\left(\frac{\lambda(L)}{\lambda(L')}\bar{\eta}_0\right)$$

A strict inequality holds if Φ is not pseudo-L-ergodic, or not pseudo-L'-ergodic. So, in a transition from $P_{L'}^0$ to P_L^0 the importance of realizations for which $\lambda(L)\bar{\eta}_0/\lambda(L')$ is relatively large, should be reconsidered:

> P_L^0 arises from $P_{L'}^0$ by first changing the weights of the realizations by way of

the weight function $\lambda(L')/(\lambda(L)\bar{\eta}_0)$, followed by shifting the origin to an L-point chosen at random from the first n L-points and letting n tend to infinity.

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